Matching with contracts: a deferred acceptance algorithm and the fullset of stable allocations

Abstract: In this paper, we present a deferred-acceptance algorithm for the model of Matching with Contracts Many-to-many with all the agents' preferences satisfying Substitutability. Therefore, we provide a constructive proof of the existence of at least one stable allocation for this model and show that it is the optimal allocation for the agents which make the offers and the worst stable allocation for the other market-side. Also, we include a proof of the fact that the set of stable allocations has lattice structure with respect to the Blair´s partial orderings for each market-side. Such lattices are dual, so there is a counterposition of interests between both market-sides. Finally, we use our deferred-acceptance algorithm and the mentioned lattice structure to calculate the complete set of stable allocations.

Keywords: contracts, deferred-acceptance, lattice, stable allocations, substitutability

JEL Codes: C78, J41

1. Introduction

The model of matching with contracts many-to-one was introduced by Hatfield and Milgrom (2005) as a generalization of the model in Kelso and Crawford (1982), the college admissions problem and ascending package auctions. Some models of matching with contracts many-to-many are studied in Klaus and Markus (2009); Hatfield and Kominers (2012); etc.

As we describe in section 2, the models with contracts many-to many, the set of agents splits into two disjoint finite subsets: Doctors and Hospitals. Every contract is bilateral, involves one doctor and one hospital. A contract should be understood as a package of conditions, which would characterize the relationship among its parts if it was signed. Each agent can sign multiple contracts but they must involve different agents in the opposite market-side, so that, every pair doctor-hospital can sign at most one contract in this model. An outcome or allocation is a set of contracts satisfying the last condition. Each agent has preferences over allocations with all the contracts involving him. Along all this work, the agents´s preferences are supposed to satisfy a condition known as substitutability. An allocation is individually rational if there is not an agent who prefers a proper subset of the set of contracts that involve him in the allocation rather than his assignment by the allocation. An individually rational allocation is stable if there is not a contract which both involved agents would prefer to add to their current set of contracts (possibly dropping some of such contracts). In this paper, we focus in the set of stable allocations in markets with contracts many-to-many with sustitutable preferences of all the agents and extend, from the context without contracts, some relevant results.

Gale and Shapley (1962) introduced a deferred-acceptance algorithm to compute a stable allocation for each market of matching one-to-one without contracts. They also proved that the matching obtained from their algorithm was the best stable allocation for one market-side and the worst stable allocation for the other market-side. For the model of matching many-to-one without contracts, Kelso and Crawford (1982) and Roth (1984) adapted the results of Gale and Shapley (1962). In section 3 of this paper, we develop two algorithms which, starting from the empty allocation as input, give respective deferred-acceptance algorithms for the model of matching with contracts many-to-many, as we explain in section 4. There, we also show that both stable allocations produced by our deferred-acceptance algorithms are, respectively, the best stable allocation for one and the other market-side (and the worst stable allocation for the opposite market-side in every case).

In previous literature, it has been proved that the sets of stable matchings have lattice structure with respect to certain partial orderings that were defined. This means that the partial orderings arrange the elements in the set so that the least upper bound (*l.u.b.*) and the greatest lower bound (*g.l.b.*) between any pair of such elements exist and belong to the same set. These results have been useful for the construction of several algorithms that yield stable matchings; and, in many cases, they have led to other important discoveries as the coincidence/counterposition of interests between agents in the same/opposite market-sides respectively.

For the model one-to-one without contracts, Knuth (1976) demonstrated that the set of all stable allocations is a lattice with respect to the unanimous partial orderings to each market-side. Roth (1985) proved that the *l.u.b.* and the *g.l.b.* proposed by Knuth (1976) did not work in a more general model of matching many-to-many. Earlier, Blair (1988) introduced a natural extension of the partial orderings used by Knuth (1976), which endowed the set of stable matchings with lattice structure in the markets of matching many-to-many with substitutable preferences; he also proved there the existence of a kind of counterposition of interests between both market-sides. The proofs in Blair (1988) can be traduced in terms of contracts, producing the mentioned results for the model in Klaus and Markus (2009), which essentially coincides with the model in Blair (1988). In section 5, we obtain such results by using our deferred acceptance algorithms to compute the corresponding *l.u.b.* and *g.l.b. between two any stable allocations*: we prove that the set of stable allocations has lattice structure with respect to the Blair´s partial orderings for each market-side and show the duality between both mentioned lattices.

In section 6, we use our deferred-acceptance algorithm and the mentioned lattice structure to produce an algorithm which calculates the full set of stable allocations for any market of matching with contracts many-to-many with sustitutable preferences. We followed the work of Martínez et al (2004) for markets without contracts, to develop our algorithm and prove that it works.

2. Preliminaries

We consider the model of matching with contracts defined in Hatfield and Kominers (2012). The markets with contracts have two disjoint sides: a finite set of doctors D and a finite set of hospitals H. The agents from one market-side have to be assigned to agents in the opposite side through contracts which state the conditions (salary, schedules, work tasks, etc.) implied by the relationship among two agents. Such conditions are not fixed beforehand unlike what happens in the markets without contracts.

There is a finite set of contracts $\mathbf{X}$. Each contract matches one doctor $x_D \in D$ with one hospital $x_H \in H$ and establishes the conditions that would characterize the relationship among its parts if they signed it. Each agent can sign at most one contract with each agent in the opposite market-side.

An allocation is a subset that contains at most one contract for each pair doctor-hospital.

Definition 1. *A set of contracts $Z \subseteq \mathbf{X}$ is an* ***allocation*** *if for all $x, y \in Z$,*

$$y_D = x_D \text{ and } y_H = x_H \text{ imply } x = y$$

Note that the empty set is an assignment.

Given $Y \subseteq \mathbf{X}$ a subset of contracts, for each $i \in D \cup H$ we denote:

$$Y_i = \{x \in Y : i \in \{x_D, x_H\}\}$$

For each agent $i \in D \cup H$, $\succ_i$ is a strict, transitive and complete binary relation over the allocations contained in $\mathbf{X}_i$. A profile of preferences P is a vector $\mathbf{P} = (P_{d_1}, ..., P_{d_{|D|}}, P_{h_1}, ..., P_{h_{|H|}})$ defining the preferences of all the agents in a market. Here, $|D|$ and $|H|$ represent the cardinalities of D and H respectively and P_i is the list of preferences of the agent i. As example, $P_h : \{x, y, z\} \succ_h \{x, y\} \succ_h \{x, z\} \succ_h \{z\} \succ_h \varnothing \succ_h \{y, z\} \succ_h \{y\} \succ_h \{x\}$ could be the list of preferences of a hospital.

Notation: $X \succeq_i Y$ means $X \succ_i Y$ or $X = Y$.

A particular market with contracts is denoted by $(\mathbf{X}, \mathbf{P})$ since it is determined by the set of all existing contracts $\mathbf{X}$ and the preferences profile $\mathbf{P}$.

If $\mathbf{X}$ contains one and only one contract for each pair doctor-hospital, we can solve the allocation problem by using an equivalent matching market without contracts and vice versa. Therefore, the markets of matching without contract are special cases of the model of matching with contracts.

Given $Y \subseteq \mathbf{X}$ a subset of contracts, we distinguish for each agent $i \in D \cup H$ the best allocation contained in Y_i according to P_i.

Definition 2. *Given a subset of contracts $Y \subseteq \mathbf{X}$, and an agent $i \in D \cup H$, the* ***choice set*** *of i given Y is:*

$$C_i(Y) = \max_{\succ_i} \{Z \subseteq Y_i : Z \textit{ is an allocation}\}$$

Note that $C_d(Y)$ could be the empty allocation.

$R_i(Y) = Y_i - C_i(Y)$ is the set of contracts that are rejected by i given the set of contracts Y.

Example 1. *Consider the list of preferences*
$P_i : \{x, y, z\} \succ_i \{x, y\} \succ_i \{x, z\} \succ_i \{z\} \succ_i \varnothing_i \succ_i \{y, z\} \succ_i \{y\} \succ_i \{x\}$.

Here $\mathbf{X}_i = \{x, y, z\}$. For $Y = \{y, z\}$ we have $C_i(Y) = \{z\}$ and $R_i(Y) = \{y\}$.□

We will use the following notation:
$C_D(Y) = \cup_{d \in D} C_d(Y)$ will be the choice set of all doctors given Y;
$C_H(Y) = \cup_{h \in H} C_h(Y)$ will be the choice set of all hospitals given Y;
$R_D(Y) = Y - C_D(Y) = \cup_{d \in D} R_d(Y)$ will be the set of all contracts belonging to Y which are rejected by doctors;

$R_H(Y) = Y - C_H(Y) = \cup_{h \in H} R_h(Y)$ will be the set of all contracts belonging to Y which are rejected by hospitals.

The following properties come from definition of the choice set. For all $Y \subseteq \mathbf{X}$ and $i \in D \cup H$:
(I) $D \cup H, C_i(Y) \subseteq Y$.
(II) $C_i(Y) \subseteq Z \subseteq Y$ *implies* $C_i(Z) = C_i(Y)$.
(III) $C_i(C_i(Y)) = C_i(Y)$.

At models of matching with contracts, the individual rationality is defined in similar way as in models without contracts. In this work we need also distinguish the individual rationality for each side of the market.

Definition 3. *The allocation $Y \subseteq \mathbf{X}$ is:*
(i) ***individually rational (IR)*** *if $C_D(Y) = C_H(Y) = Y$*
(ii) ***individually rational for doctors (IRD)*** *if $C_D(Y) = Y$*
(iii) ***individually rational for hospitals (IRH)*** *if $C_H(Y) = Y$.*

The concept of stability is an adaptation to this context of the pairwise stability utilized in models without contracts. See Blair (1988).

Definition 4. *Given an allocation $Y \subseteq \mathbf{X}$, the contract $x \in \mathbf{X} \backslash Y$ is a* ***blocking contract*** *for Y if*

$$x \in C_{x_D}(Y \cup \{x\}) \cap C_{x_H}(Y \cup \{x\})$$

Definition 5. *The allocation $Y \subseteq \mathbf{X}$ is a* ***stable allocation*** *if*
(i) Y is individually rational;
(ii) There are no blocking contracts for Y.

Given a market with contracts $(\mathbf{X}, \mathbf{P})$, we denote as $S(\mathbf{X}, \mathbf{P})$ the set of all stable allocations in this market.

For models without contracts it has been proved that $S(\mathbf{X}, \mathbf{P})$ is nonempty if the preferences of all agents satisfy the condition of substitutability, which states that the agents do not consider as complementary the agents in the opposite market-side. Hatfield and Milgrom (2005) introduce an extension of substitutability for models with contracts and Hatfield and Kominers (2012) prove the existence of at least one stable allocation for the model of matching with contracts many to many where all the agents' preferences satisfy such condition.
In models with contracts the preferences of an agent satisfy substitutability if no contract stops being chosen because another contract stops being available. This is, the agents do not consider the contracts as complementary among themselves.

Definition 6. *Preferences of agent $i \in D \cup H$ satisfy* ***substitutability*** *if*

$$R_i(X) \subseteq R_i(Y)$$

for all sets X, Y such that $X_i \subseteq Y_i \subseteq \mathbf{X}$.

Assuming that preferences of agent $i \in D \cup H$ satisfy substitutability, following Blair (1988) we can justify the next additional properties of the choice set for all $X, Y \subseteq \mathbf{X}$.

(IV) $C_i(X \cup Y) \cap X \subseteq C_i(X)$.In fact, a contract belonging to X which is chosen by i among the contracts in $X \cup Y$, due to the substitutability of preferences of i, it also will be chosen among contracts belonging to the smaller set X.

(V) $C_i(X \cup Y) = C_i(C_i(X) \cup Y)$. In fact, every contract in X which belongs to $C_i(X \cup Y)$, also belongs to $C_i(X)$ because of substitutability, therefore the choice set of i given $X \cup Y$ will coincide with the choice set of i given $C_i(X) \cup Y$.

(VI) If $X \subseteq Y$, $C_i(Y) - X \subseteq C_i(Y - X)$. In fact, the contracts which do not belong to X and belong to the choice set of i given Y, due to the substitutability of preferences of i, will belong to the choice set of i given the smaller set $Y - X$.

From Blair (1988) we can obtain a proof of the fact that the nonempty set of all stable allocations has a lattice structure with respect to the partial orderings defined by him in such paper. This means that if the stable allocations are arranged according to each of these partial orderings, we can find the least upper bound and the greatest lower bound among two any stable allocations. Later, we will use the algorithms that we are going to define in the next section to give an alternative proof of this fact. Next, we include the formal definition of lattice.

Definition 7. *the set M has a lattice structure with respect to the partial ordering $\geq$ if there exist two binary operations $\vee$ and $\wedge$ on M such that for all $x, y, z \in M$ the following properties hold:*
$x \vee y \in M$
$x \wedge y \in M$
$x \vee y \geq x$ *and* $x \vee y \geq y$
$x \geq x \wedge y$ *and* $y \geq x \wedge y$
$z \geq x$ *and* $z \geq y$ *imply* $z \geq x \vee y$
$x \geq z$ *and* $y \geq z$ *imply* $x \wedge y \geq z$

The first and second properties say that $\vee$ and $\wedge$ binary operations on M. The remaining conditions say that $x \vee y$ and $x \wedge y$ are, respectively, the least upper bound and the great lower bound of x and y according to $\geq$. The quadruple $(M, \geq, \vee, \wedge)$ is called a lattice on M.

Along this work we will use the following partial orderings to study topics related to lattice structures.

Definition 8. *The allocation Y is unanimously most preferred by the doctors to the allocation Z ($Y \succeq_D Z$) if for all $d \in D$, Y_d is strictly preferred by d to Z_d or $Y_d = Z_d$. This is the* ***unanimous partial order for doctors*** .

Definition 9. *The allocation Y is unanimously most preferred by the hospitals to the allocation Z ($Y \succeq_H Z$) if for all $h \in H$, Y_h is strictly preferred by h to Z_h or $Y_h = Z_h$. This is the* ***unanimous partial order for hospitals*** .

Definition 10. *The allocation Y is preferred by the doctors to the allocation Z according to Blair ($Y \succeq_D^B Z$) if for all $d \in D$, $C_d(Y \cup Z) = Y_d$. This is the* ***Blair's partial ordering for doctors*** .

Definition 11. *The allocation Y is preferred by the hospitals to the allocation Z according to Blair ($Y \succeq_H^B Z$) if for all $h \in H$, $C_h(Y \cup Z) = Y_h$. This is the* ***Blair's partial ordering for hospitals*** .

REMARK 1. *$Y \succeq_D^B Z$ implies $Y \succeq_D Z$; and $Y \succeq_H^B Z$ implies $Y \succeq_H Z$.*

To develop our algorithms, we need to distinguish the following sets of contracts. Given $Y \subseteq \mathbf{X}$, $d \in D$ and $h \in H$:

$$I(d, Y) = \left\{ y \in \mathbf{X}_d : \ y \in C_{y_H}(Y \cup \{y\}) \right\}$$

and

$$I(h, Y) = \left\{ y \in \mathbf{X}_h : y \in C_{y_D}(Y \cup \{y\}) \right\}$$

Moreover, we will denote

$$I(D, Y) = \underset{d \in D}{\cup} I(d, Y)$$

$$I(H, Y) = \underset{h \in H}{\cup} I(h, Y)$$

DEFINITION 12. *Let IP^H and IP^D be the families of sets of contracts which satisfy, respectively, the following inclusion properties:*

$$IP^D = \{Y \subseteq \mathbf{X} / Y \subseteq C_D\left(I\left(D, Y\right)\right)\}.$$

$$IP^H = \{Y \subseteq \mathbf{X} / Y \subseteq C_H\left(I\left(H, Y\right)\right)\}$$

The following properties will be very useful in the development of this work

LEMMA 1. *Given two sets of contracts $Y, Z \subseteq \mathbf{X}$:*
i) $Y \subseteq Z$ implies $I(D, Z) \subseteq I(D, Y)$ and $I(H, Z) \subseteq I(H, Y)$.
ii) $I(d, C_H(Z)) = I(d, Z)$ for every $d \in D$ and $I(h, C_D(Z)) = I(h, Z)$ for every $d \in D$
iii) If Y is an IRH allocation, then $Y \in IP^D$; if Y is an IRD allocation, then $Y \in IP^H$.

PROOF. i) Suppose $y \in I(y_D, Z)$, then $y \in C_{y_H}(Z \cup \{y\})$ and consequently $y \in C_{y_H}(Y \cup \{y\})$ since y_H has substitutable preferences. Thus $y \in I(y_D, Y)$. Therefore, $I(D, Z) \subseteq I(D, Y)$. The proof of $I(H, Z) \subseteq I(H, Y)$ is analogous.
ii) Given $d \in D$, we have $I(d, C_H(Z)) = \{y \in \mathbf{X} : y \in C_{y_H}(C_H(Z) \cup \{y\})\}$. But $C_{y_H}(C_H(Z) \cup \{y\}) = C_{y_H}(C_{y_H}(Z) \cup \{y\}) = C_{y_H}(Z \cup \{y\})$ and consequently $\{y \in \mathbf{X} : y \in C_{y_H}(C_H(Z) \cup \{y\})\} = \{y \in \mathbf{X} : y \in C_{y_H}(Z \cup \{y\})\} = I(d, Z)$. So, $I(d, C_H(Z)) = I(d, Z)$. The proof of $I(h, C_D(Z)) = I(h, Z)$ is analogous.
iii) Given $d \in D$, we have $Y_d \subseteq I(d, Y)$ because Y is an IRH allocation. This fact together with the property (IV) of the choice sets imply $C_d(I(d, Y)) \cap Y_d = C_d(I(d, Y) \cup Y_d) \cap Y_d \subseteq C_d(Y_d) = Y_d$. As a consequence, $Y_d \subseteq C_d(I(d, Y)) \subseteq C_D\left(I\left(D, Y\right)\right)$. Then, $Y \in IP^D$. Analogously can be proved that. $Y \in IP^H$ for every IRD allocation Y. $\square$

3. Doctors Offering Algorithm and Hospitals Offering Algorithm

In this section, we will develop a constructive demonstration of the existence of at least one stable allocation for each market of matching with contracts many-to-many where the agents' preferences satisfy substitutability. For this purpose, we will present two algorithms: the Doctors Offering Algorithm and the Hospitals Offering Algorithm, and we will show their convergence to stable allocations whenever they start from an IRH allocation or an IRD allocation respectively. Since the empty allocation is IR, we can apply to it any of such algorithms to obtain a stable allocation.

Doctors Offering Algorithm (DOA)

Input:
A model of contracts $(\mathbf{X}, P)$ and an IRH allocation $Y \subseteq \mathbf{X}$.
Begin:
$X^0 = Y$ and $i := 0$.
Repeat:
Step 1: Determine $I\left(D, X^i\right)$
Step 2: calculate $C_D\left(I\left(D, X^i\right)\right)$
Step 3: Define $X^{i+1} = C_H\left(C_D\left(I\left(D, X^i\right)\right)\right)$
If $X^{i+1} = X^i$, the algorithm stops with output X^{i+1}.
If $X^{i+1} \neq X^i$, define $i := i+1$ and repeat steps 1 to 3
End

The next example describes the application of the DOA, starting from the empty allocation, in a particular market.

Example 2. *Consider a market* $(\mathbf{X}, P)$ *such that* $\mathbf{X} = \{x_{11}, x_{12}, x_{21}, y_{21}, x_{32}, y_{32}\}$ *where* x_{ij} *and* y_{ij} *denote two different contracts involving doctor* d_i *and hospital* h_j*, and the profile of preferences* P *is the following:*
$P_{d_1} : \{x_{11}, x_{12}\} \succ_{d_1} \{x_{11}\} \succ_{d_1} \{x_{12}\} \succ_{d_1} \varnothing$
$P_{d_2} : \{y_{21}\} \succ_{d_2} \{x_{21}\} \succ_{d_2} \varnothing$
$P_{d_3} : \{y_{32}\} \succ_{d_3} \{x_{32}\} \succ_{d_3} \varnothing$
$P_{h_1} : \{x_{11}, x_{21}\} \succ_{h_1} \{x_{11}, y_{21}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{x_{21}\} \succ_{h_1} \{y_{21}\} \succ_{h_1} \varnothing$
$P_{h_2} : \{x_{32}\} \succ_{h_2} \{y_{32}\} \succ_{h_2} \varnothing$

Doctors offering Algorithm:
Input
The market $(\mathbf{X}, P)$ *and the IRH allocation* $\varnothing$.
Begin:
$Y^0 = \varnothing$
Iterations
$\mathbf{i = 1}$
Determine

$$I(D, Y^0) = \{x_{11}, x_{21}, y_{21}, x_{32}, y_{32}\}, \text{ \textit{in fact:}}$$

$x \in C_{x_H}(\{x\} \cup \varnothing)$ *for all* $x \in \mathbf{X} - \{x_{12}\}$ *whereas* $C_{h_2}(\{x_{12}\} \cup \varnothing) = \varnothing$.
calculate

$$C_D\left(I(D, Y^0)\right) = \{x_{11}, y_{21}, y_{32}\} \text{ \textit{in fact:}}$$

$C_{d_1}(I(d_1, Y^0)) = C_{d_1}(\{x_{11}\}) = \{x_{11}\}$
$C_{d_2}(I(d_2, Y^0)) = C_{d_2}(\{x_{21}, y_{21}\}) = \{y_{21}\}$
$C_{d_3}(I(d_3, Y^0)) = C_{d_3}(\{x_{32}, y_{32}\}) = \{y_{32}\}$
Define

$$Y^1 = C_H(C_D\left(I(D, Y^0)\right)) = \{x_{11}, y_{21}, y_{32}\} \text{ \textit{in fact:}}$$

$C_{h_1}(C_D\left(I(D, Y^0)\right)) = C_{h_1}(\{x_{11}, y_{21}\}) = \{x_{11}, y_{21}\}$

$C_{h_2}(C_D\left(I(D,Y^0)\right)) = C_{h_2}(\{y_{32}\}) = \{y_{32}\}$
Since $Y^1 \neq Y^0$, we realize a new iteration.
$\mathbf{i = 2}$
Determine

$$I(D,Y^1) = \{x_{11}, x_{21}, y_{21}, x_{32}, y_{32}\}\,,\ \textit{in fact:}$$

$x_{11} \in C_{h_1}\left(\{x_{11}\} \cup Y^1\right) = \{x_{11}, y_{21}\}\,;$
$x_{12} \notin C_{h_1}\left(\{x_{12}\} \cup Y^1\right) = \{x_{11}, y_{21}\}\,;$
$x_{21} \in C_{h_1}\left(\{x_{21}\} \cup Y^1\right) = \{x_{11}, x_{21}\}\,;$
$y_{21} \in C_{h_1}\left(\{y_{21}\} \cup Y^1\right) = \{x_{11}, y_{21}\}\,;$
$x_{32} \in C_{h_2}\left(\{x_{32}\} \cup Y^1\right) = \{x_{32}\}\,;$
$y_{32} \in C_{h_2}\left(\{y_{32}\} \cup Y^1\right) = \{y_{32}\}$
Since $I(D,Y^1) = I(D,Y^0)$, we obtain

$$C_D\left(I(D,Y^1)\right) = C_D\left(I(D,Y^0)\right) = \{x_{11}, y_{21}, y_{32}\}$$

and

$$Y^2 = C_H(C_D\left(I(D,Y^1)\right)) = C_H(C_D\left(I(D,Y^0)\right)) = \{x_{11}, y_{21}, y_{32}\}$$

The algorithm stops because $Y^2 = Y^1 = \{x_{11}, y_{21}, y_{32}\}$.
Output
$\{x_{11}, y_{21}, y_{32}\}$ □

To demonstrate that the previous algorithm stops, first we will show that the allocation X^i, obtained at the end of the iteration i, satisfies $X^i \subseteq C_D\left(I\left(D, X^i\right)\right)$, this is, $X^i \in IP^D$ for all i.

Lemma 2. *Suppose that the preferences of all the agents satisfy substitutability. Let X^i the allocation obtained at the end of the iteration i of the DOA applied from an IRH allocation. Then X^i is IR and consequently $X^i \in IP^D$ and $X^i \in IP^H$ for every i.*

Proof. Given $i \geq 1$, since $X^i = C_H\left(C_D\left(I\left(D, X^{i-1}\right)\right)\right)$, then $C_H\left(X^i\right) = C_H\left[C_H\left(C_D\left(I\left(D, X^{i-1}\right)\right)\right)\right] = C_H\left(C_D\left(I\left(D, X^{i-1}\right)\right)\right) = X^i$, so X^i is IRH.
Moreover, given $x \in X^i = C_H\left(C_D\left(I\left(D, X^{i-1}\right)\right)\right)$, we have $x \in C_{x_D}\left(I\left(D, X^{i-1}\right)\right)$; and $x \in C_{x_D}\left(C_H\left(C_D\left(I\left(D, X^{i-1}\right)\right)\right)\right) = C_{x_D}\left(X^i\right)$ because $C_H\left(C_D\left(I\left(D, X^{i-1}\right)\right)\right) \subseteq I\left(D, X^{i-1}\right)$ and the hypothesis of substitutability. So, $X^i \subseteq C_D\left(X^i\right)$ which implies that X^i is IRD.
Therefore, $X^i \in IP^D$ and $X^i \in IP^H$ according to lemma 1 (iii). □

Remark 2. *DOA stops because in every iteration all the hospitals improve weakly and at least one of them improves strictly if $X^{i+1} \neq X^i$.*
in fact, since $X^i \in IP^D$, this is, $X^i \subseteq C_D\left(I\left(D, X^i\right)\right)$ for every i, then for each $h \in H$ we obtain

$$X_h^{i+1} = C_h\left(C_D\left(I\left(D, X^i\right)\right)\right) = C_h\left(X^i \cup C_D\left(I\left(D, X^i\right)\right)\right) \succeq_h C_h\left(X^i\right) = X_h^i$$

Where the last equality is due to the fact that X^i is an IRH allocation for every $i \geq 0$.

To prove that the allocation obtained as result of applying the DOA to an IRH assignment is stable, we will need the following lemma.

LEMMA 3. *Let $Z \subseteq \mathbf{X}$ be an IRH allocation such that $Z = C_D(I(D,Z))$. Then, Z is stable.*

PROOF. Because of the hypothesis Z is IRH; moreover, Z is IRD since $C_D(Z) = C_D(C_D(I(D,Z))) = C_D(I(D,Z)) = Z$. Thus Z is IR.
Suppose the existence of a blocking contract for Z, this is, a contract $y \notin Z$ such that $y \in C_{y_D}(Z \cup \{y\}) \cap C_{y_H}(Z \cup \{y\})$. Then, $y \in C_{y_D}(Z \cup \{y\})$ implies $C_{y_D}(Z \cup \{y\}) \succ_{y_D} C_{y_D}(Z)$. Since Z is IR, $Z_d \subseteq C_d(I(d,Z)) \subseteq I(d,Z)$ for all $d \in D$ according to lemma 1 (iii) , so $Z \subseteq I(D,Z)$. Moreover, $y \in C_{y_H}(Z \cup \{y\})$ because y is a blocking contract for Z. Then $y \in I(y_D, Z)$. Therefore $Z \cup \{y\} \subseteq I(D,Z)$. Consequently, $C_{y_D}(I(D,Z)) \succeq_{y_D} C_{y_D}(Z \cup \{y\}) \succ_{y_D} C_{y_D}(Z) = Z_{y_D}$. Thus, $Z_{y_D} \neq C_{y_D}(I(D,Z))$ contradicting $Z = C_D(I(D,Z))$. Therefore, Z is a stable allocation. □

LEMMA 4. *Let X^{i+1} the outcome of the DOA applied from a IRH allocation. Then, $X^{i+1} = C_D\left(I\left(D, X^{i+1}\right)\right)$.*

PROOF. We will prove the double inclusion.
On the one hand $X^{i+1} \subseteq C_D\left(I\left(D, X^{i+1}\right)\right)$ because, according to lemma 2, $X^{i+1} \in IP^D$.
On the other hand, $z \in C_D\left(I\left(D, X^{i+1}\right)\right)$ implies $z \in I\left(D, X^{i+1}\right)$ which means that $z \in C_{z_H}\left(\{z\} \cup X^{i+1}\right) = C_{z_H}\left(\{z\} \cup C_H\left(C_D\left(I\left(D, X^i\right)\right)\right)\right) = C_{z_H}\left(\{z\} \cup C_D\left(I\left(D, X^i\right)\right)\right) = C_{z_H}\left(C_D\left(I\left(D, X^{i+1}\right)\right)\right) \subseteq C_H\left(C_D\left(I\left(D, X^{i+1}\right)\right)\right) = C_H\left(C_D\left(I\left(D, X^i\right)\right)\right) = X^{i+1}$.
Therefore, $C_D\left(I\left(D, X^{i+1}\right)\right) \subseteq X^{i+1}$. The penultimate inequality is due to $X^{i+1} = X^i$. □

THEOREM 1. *Let X^{i+1} the outcome of the DOA applied from a IRH allocation. Then, X^{i+1} is stable.*

PROOF. This theorem is a immediate consequence of lemmas 3 and 4. □

Next, we prove constructively the existence of a stable allocation.

THEOREM 2. *Let $(\mathbf{X}, P)$ be a market with contracts where the preferences of all the agents satisfy substitutability. Then, the set of stable allocations $S(\mathbf{X}, P)$ is nonempty in such market..*

PROOF. The allocation $X^0 = \varnothing$ is IRH. Therefore, we obtain a stable allocation if we apply the DOA starting from $X^0 = \varnothing$. □

We define the following algorithm where the roles among both sides of the market are exchanged.

Hospitals Offering Algorithm (HOA)

Input:
A model of contracts $(\mathbf{X}, P)$ and an IRD allocation $Y \subseteq \mathbf{X}$.

Begin:
$X^0 = Y$ and $i := 0$.

Repeat:
Step 1: Determine $I\left(H,X^{i}\right)$
Step 2: Calculate $C_{H}\left(I\left(H,X^{i}\right)\right)$
Step 3: Define $X^{i+1}=C_{D}\left(C_{H}\left(I\left(H,X^{i}\right)\right)\right)$
If $X^{i+1}=X^{i}$, the algorithm stops with output X^{i+1}.
If $X^{i+1}\neq X^{i}$, define $i:=i+1$ and repeat steps 1 to 3
End.

The demonstration of the fact that the previous algorithm produces a stable allocation as outcome is analogous to the corresponding proof for the DOA. We only need to exchange the market-sides.

In the next example, the HOA is applied from the IRD allocation $\varnothing$ in the same market introduced in example 2. It has illustrative purposes and will be useful later.

EXAMPLE 3. *Consider the market* $(\mathbf{X},P)$ *introduced in example 2*

Hospitals offering Algorithm:
Input
The market $(\mathbf{X},P)$ *and the IRD allocation* $\varnothing$.
Begin
$Y^{0}=\varnothing$
Iterations
$\mathbf{i=1}$
Determine

$$I(H,Y^{0})=\mathbf{X} \textit{ in fact:}$$

$x\in C_{x_D}\left(\{x\}\cup\varnothing\right)$ *for all* $x\in\mathbf{X}$.
Calculate

$$C_{H}\left(I(H,Y^{0})\right)=\{x_{11},x_{21},x_{32}\} \textit{ in fact:}$$

$C_{h_1}(I(h_1,Y^{0}))=C_{h_1}(\{x_{11},x_{21},y_{21}\})=\{x_{11},x_{21}\}$
$C_{h_2}(I(h_2,Y^{0}))=C_{h_2}(\{x_{12},x_{32},y_{32}\})=\{x_{32}\}$
Define

$$Y^{1}=C_{D}(C_{H}\left(I(H,Y^{0})\right))=\{x_{11},x_{21},x_{32}\} \textit{ in fact:}$$

$C_{d_1}(C_{H}\left(I(H,Y^{0})\right))=C_{d_1}(\{x_{11}\})=\{x_{11}\}$
$C_{d_2}(C_{H}\left(I(H,Y^{0})\right))=C_{d_2}(\{x_{21}\})=\{x_{21}\}$
$C_{d_3}(C_{H}\left(I(H,Y^{0})\right))=C_{d_3}(\{x_{32}\})=\{x_{32}\}$
Since $Y^{1}\neq Y^{0}$, *we realize a new iteration.*
$\mathbf{i=2}$
Determine

$$I(H,Y^{1})=\mathbf{X}, \textit{ in fact:}$$

$x_{11}\in C_{d_1}\left(\{x_{11}\}\cup Y^{1}\right)=\{x_{11}\}$;
$x_{12}\in C_{d_1}\left(\{x_{12}\}\cup Y^{1}\right)=\{x_{11},x_{12}\}$;
$x_{21}\in C_{d_2}\left(\{x_{21}\}\cup Y^{1}\right)=\{x_{21}\}$;
$y_{21}\in C_{d_2}\left(\{y_{21}\}\cup Y^{1}\right)=\{y_{21}\}$;
$x_{32}\in C_{d_3}\left(\{x_{32}\}\cup Y^{1}\right)=\{x_{32}\}$;

$y_{32} \in C_{d_3}\left(\{y_{32}\} \cup Y^1\right) = \{y_{32}\}$.
Since $I(H, Y^1) = I(H, Y^0)$, *we obtain*

$$C_H\left(I(H, Y^1)\right) = C_H\left(I(H, Y^0)\right) = \{x_{11}, x_{21}, x_{32}\}$$

and

$$Y^2 = C_D(C_H\left(I(H, Y^1)\right)) = C_D(C_H\left(I(H, Y^0)\right)) = \{x_{11}, x_{21}, x_{32}\}$$

The algorithm stops because $Y^2 = Y^1 = \{x_{11}, x_{21}, x_{32}\}$.
Output
$\{x_{11}, x_{21}, x_{32}\}$ □

4. Optimal Allocations

Next theorem proves the existence of a counterposition of interests between both market-sides relative to the Blair´s partial orderings.

THEOREM 3. *If* $Y \subseteq \mathbf{X}$ *and* $Z \subseteq \mathbf{X}$ *are two stable allocations, then:* $Y \succeq_H^B Z$ *if and only if* $Z \succeq_D^B Y$.

PROOF. Suppose that $Y \succeq_H^B Z$ but $Z \succeq_D^B Y$ is not fulfilled, this is, there exists $d \in D$ such that $C_d(Y \cup Z) \neq Z_d$. Then, since d has substitutable preferences and Z is an IR allocation, a contract $y \in Y_d - Z$ such that $y \in C_d(\{y\} \cup Z)$ must exist. Moreover, $y \in C_{y_H}(\{y\} \cup Z)$ because y_H has substitutable preferences and $Y \succeq_H^B Z$ implies $Y_{y_H} = C_{y_H}(Y \cup Z)$. Thus, the stability of Z is contradicted. Therefore $Z \succeq_D^B Y$.
The fact that $Z \succeq_D^B Y$ implies $Y \succeq_H^B Z$ has an analogous proof.. □

Notation: Given an IRH allocation $Y \subseteq \mathbf{X}$, we will denote $D^O(Y)$ the stable allocation obtained as output when the DOA is applied starting from the allocation Y.
Analogously, given an IRD allocation $Y \subseteq \mathbf{X}$, we will denote $H^O(Y)$ the stable allocation obtained as output when the HOA is applied starting from the allocation Y.
The following lemma will be a tool to prove that the stable allocations $D^O(\varnothing)$ and $H^O(\varnothing)$ are the stable allocations unanimously more preferred by the doctors and unanimously more preferred by the hospitals respectively.

LEMMA 5. *Let* $Z \subseteq \mathbf{X}$ *be a stable allocation and suppose that the preferences of all the agents satisfy substitutability.*
i) Assume $Y \in IP^D$. *Then,* $Z \succeq_H^B Y$ *if and only if* $Z \succeq_H^B D^O(Y)$.
ii) Assume $Y \in IP^H$. *Then,* $Z \succeq_D^B Y$ *if and only if* $Z \succeq_D^B H^O(Y)$.

PROOF. i) Let $Y = Y^0, Y^1, ..., Y^k = D^O(Y)$. Given $i \in \{0, ..., k-1\}$, we have proved in lemma 2 that $Y^i \in IP^D$, this is, $Y^i \subseteq C_D\left(I\left(D, Y^i\right)\right)$. Moreover $Y^{i+1} = C_H\left(C_D\left(I\left(D, Y^i\right)\right)\right)$, then $C_H(Y^i \cup Y^{i+1}) = C_H(Y^i \cup C_H\left(C_D\left(I\left(D, Y^i\right)\right)\right)) = C_H(Y^i \cup C_D\left(I\left(D, Y^i\right)\right)) = C_H(C_D\left(I\left(D, Y^i\right)\right)) = Y^{i+1}$. So, $Y^{i+1} \succeq_H^B Y^i$ for every $i = 0, ..., k-1$. Consequently, $D^O(Y) = Y^k \succeq_H^B Y^0 = Y$ since $\succeq_H^B$ is transitive.
Then, If we assume that $Z \succeq_H^B D^O(Y)$, we obtain $Z \succeq_H^B D^O(Y) \succeq_H^B Y$.

Inversely, under the hypothesis $Z \succeq_H^B Y = Y^0$, we will prove that $Z \succeq_H^B Y^i$ implies $Z \succeq_H^B Y^{i+1}$ for every $i = 0, 1, ...k-1$. This will allow us to conclude inductively that $Z \succeq_H^B Y^k = D^O(Y)$.
Given $i \in \{0, ..., k-1\}$, suppose $Z \succeq_H^B Y^i$ so that $C_H(Z \cup Y^i) = Z$. Because of substitutability, the later implies $z \in C_{z_H}(Y^i \cup \{z\})$ for every $z \in Z$. Thus $Z \subseteq I(D, Y^i)$. Therefore

$$C_D I(D, Y^i) \subseteq I(H, Z) \qquad (I)$$

in fact: $x \in C_D\left(I(D, Y^i)\right) = C_D\left(I(D, Y^i) \cup Z \cup \{x\}\right)$ implies $x \in C_{x_D}(Z \cup \{x\})$ due to substitutability.
If we prove that $C_H(Y^{i+1} \cup Z) \subseteq Z$, then $C_H(Y^{i+1} \cup Z) \subseteq Z \subseteq Y^{i+1} \cup Z$ would imply $C_H(Y^{i+1} \cup Z) = C_H(Z) = Z$ and the proof would be completed
According to lemma 2 $Y^i \in IP^D$, this is, $Y^i \subseteq C_D\left(I(D, Y^i)\right)$. Set $S^i = C_D\left(I(D, Y^i)\right) - Y^i$. Then $C_H(Y^{i+1} \cup Z) = C_H(C_H\left(C_D\left(I\left(D, Y^i\right)\right)\right) \cup Z) = C_H(C_D\left(I\left(D, Y^i\right)\right) \cup Z) = C_H(S^i \cup Y^i \cup Z)$. So:

$$C_H(Y^{i+1} \cup Z) = C_H(S^i \cup Y^i \cup Z) \qquad (II)$$

Given a contract $y \in C_H(Y^{i+1} \cup Z)$, we have to prove that $y \in Z$. For this purpose, we will analyze separately the cases $y \in S^i$ and $y \notin S^i$.
a) Suppose $y \notin S^i$. Substitutability and (II) imply $y \in C_H(Y^i \cup Z) = Z$.
b) Suppose $y \in S^i$. Then $y \in C_D\left(I(D, Y^i)\right)$ and consequently $y \in I(H, Z)$ according to (I). This is to say that $y \in C_{y_D}(Z \cup \{y\})$. Moreover, $y \in C_{y_H}(Z \cup \{y\})$ due to substitutability. Then, $y \in Z$ because in opposite case y would be a blocking contract for the stable allocation Z, which is a contradiction. □

THEOREM 4. *Let $(\mathbf{X}, P)$ be a market with contracts where all the agents have substitutable preferences. Then,*

$$H^O(\varnothing) \succeq_H^B Y \succeq_H^B D^O(\varnothing) \text{ and } D^O(\varnothing) \succeq_D^B Y \succeq_D^B H^O(\varnothing).$$

for every stable allocation $Y \subseteq \mathbf{X}$

PROOF. Let $Y \subseteq \mathbf{X}$ be a stable allocation. Since $\varnothing \in IP^D$, $\varnothing \in IP^H$, $Y \succeq_H^B \varnothing$ and $Y \succeq_D^B \varnothing$; according to lemma 5 we have $Y \succeq_H^B D^O(\varnothing)$ and $Y \succeq_D^B H^O(\varnothing)$.
Then, due to the counterposition of interests proved in theorem 3, we also obtain $H^O(\varnothing) \succeq_H^B Y$ and $D^O(\varnothing) \succeq_D^B Y$. □

Given two stable allocations $Y, Z \subseteq \mathbf{X}$ such that $Y \succeq_H^B Z$, we have already mentioned that $Y \succeq_H Z$; analogously $Y \succeq_D^B Z$, implies $Y \succeq_D Z$. As a consequence, theorem 4 implies that $H^O(\varnothing)$ is the stable allocation unanimously most preferred by the hospitals and the stable allocation unanimously least preferred by the doctors; whereas $D^O(\varnothing)$ is the stable allocation unanimously most preferred by the doctors and the stable allocation unanimously least preferred by the hospitals. Formally,

COROLLARY 1. *Let $(\mathbf{X}, P)$ be a market with contracts where all the agents have substitutable preferences. Then,*

$$H^O(\varnothing) \succeq_H Y \succeq_H D^O(\varnothing) \text{ and } D^O(\varnothing) \succeq_D Y \succeq_D H^O(\varnothing).$$

for every stable allocation $Y \subseteq \mathbf{X}$

Remark 3. *The application of the DOA starting from the allocation $\varnothing$, constitutes an extension of the algorithm of deferred acceptance introduced by Gale and Shapley (1962) to the model with contracts; in such algorithm the doctors offer and the hospitals accept or reject offers. The steps can be described as follows: bearing in mind the set of all the contracts that are acceptable to the involved hospital, each doctor selects the first subset of such set according to his list of preferences and offer them to the corresponding hospitals. Then, every hospital considers the set of all the offers that it has just received and chooses the subset of received offers that it likes best according to its preferences. In the following iteration of the algorithm, the doctors bear in mind only the contracts that the hospitals would want to sign if they had the contracts that they accepted in the previous iteration also available (observe that all the contracts that were chosen by a hospital in the previous iteration satisfy this requirement, whereas the contracts that were rejected by the involved hospital in a previous iteration do not satisfy the requirement, so they cannot be offered again); then, each doctor selects its most preferred subset of the just described set and offers the contracts belonging to such subset to the corresponding hospitals; next, the hospitals choose their most preferred subset of just received offers and the iteration ends. The algorithm stops when all the hospitals accept exactly the same contracts in two consecutive iterations.*
The application of the HOA starting from the allocation $\varnothing$ also extends the algorithm of deferred acceptance introduced by Gale and Shapley (1962) to the model with contracts, but in this case the hospitals offer and the doctors accept or reject offers.

5. Lattice Structure

Let $(\mathbf{X}, P)$ be a market of matching with contracts many-to-many where the preferences of all the agents satisfy substitutability. In this section we will prove that the set of all the stable allocations $S(\mathbf{X}, P)$ is a lattice with respect to the Blair's partial ordering for hospitals $\succeq_H^B$. Symmetrically, it is possible to show that $S(\mathbf{X}, P)$ is a lattice with respect to the Blair's partial ordering for doctors $\succeq_D^B$; and because of the counterposition of interests relative to the Blair´s partial orderings that we have shown in theorem 3, we can conclude that both lattices are dual.

According to the definition of lattice included in the preliminaries of this work, given two stable allocations Z and Y, we have to find the stable allocations $Y \vee_H Z$ and $Y \wedge_H Z$ which are, respectively, the least upper bound and the greatest lower bound for Z and Y relative to $\succeq_H^B$. We will demonstrate that the above mentioned bounds exist and can be calculated using the algorithms DOA and HOA. For any couple of stable assignments Z and Y, we define:

$$Y \vee_H Z = D^O(C_H(Y \cup Z))$$

and

$$Y \wedge_H Z = H^O(C_D(Y \cup Z)).$$

The following lemma demonstrates that the operations $\vee_H$ and $\wedge_H$ are closed in $S(\mathbf{X}, P)$.

Lemma 6. *Let $Y \subseteq \mathbf{X}$ and $Z \subseteq \mathbf{X}$ be two stable allocations, Suppose that the preferences of all the agents satisfy substitutability. Then, $D^O(C_H(Y \cup Z))$ and $H^O(C_D(Y \cup Z))$ are stable allocations.*

Proof. $C_H(Y\cup Z)$ is an IRH allocation and $C_D(Y\cup Z)$ is an IRD allocation, therefore $D^O(C_H(Y\cup Z))$ and $H^O(C_D(Y\cup Z))$ are stable allocations. $\square$

Next lemma proves that $\vee_H$ and $\wedge_H$ calculate, respectively, the least upper bound and the greatest lower bound, relative to $\succeq_H^B$, for two any stable allocations.

Lemma 7. *Let $Y\subseteq \mathbf{X}$ and $Z\subseteq \mathbf{X}$ be two stable allocations. Then,*
i) $D^O(C_H(Y\cup Z))\succeq_H^B Y$ and $D^O(C_H(Y\cup Z))\succeq_H^B Z$.
ii) If $W\subseteq \mathbf{X}$ is a stable allocation such that $W\succeq_H^B Y$ and $W\succeq_H^B Z$, then $W\succeq_H^B D^O(C_H(Y\cup Z))$.
iii) $Y\succeq_H^B H^O(C_D(Y\cup Z))$ and $Z\succeq_H^B H^O(C_D(Y\cup Z))$.
iv) If $W\subseteq \mathbf{X}$ is a stable allocation such that $Y\succeq_H^B W$ and $Z\succeq_H^B W$, then $H^O(C_D(Y\cup Z))\succeq_H^B W$.

Proof. i) Let $C_H(Y\cup Z)=X^0,...,X^k=D^O(C_H(Y\cup Z))$ be the succession of allocations that is obtained through the DOA starting from the allocation $C_H(Y\cup Z)$. We will prove inductively that $C_H\left(Y\cup X^i\right)=X^i$ for every $i\in\{0,...,k\}$.
First, $C_H\left(Y\cup X^0\right)=C_H(Y\cup C_H(Y\cup Z))=C_H(Y\cup Y\cup Z)=C_H(Y\cup Z)=X^0$.
Supposing that $C_H\left(Y\cup X^i\right)=X^i$, we have to prove that $C_H\left(Y\cup X^{i+1}\right)=X^{i+1}$. According to lemma 2, $X^i\in IP^D$, this is, $X^i\subseteq C_D\left(I\left(D,X^i\right)\right)$. Consequently, $C_H\left(Y\cup X^{i+1}\right)=C_H\left(Y\cup C_H\left(C_D\left(I\left(D,X^i\right)\right)\right)\right)=C_H\left(Y\cup C_D\left(I\left(D,X^i\right)\right)\right)=C_H\left(Y\cup X^i\cup C_D\left(I\left(D,X^i\right)\right)\right)=C_H\left(C_H\left(Y\cup X^i\right)\cup C_D\left(I\left(D,X^i\right)\right)\right)=C_H\left(X^i\cup C_D\left(I\left(D,X^i\right)\right)\right)=C_H\left(C_D\left(I\left(D,X^i\right)\right)\right)=X^{i+1}$.
In particular, $C_H\left(Y\cup X^k\right)=X^k$, this is, $D^O(C_H(Y\cup Z))\succeq_H^B Y$.
Analogously can be proved that $D^O(C_H(Y\cup Z))\succeq_H^B Z$.
ii) $W\succeq_H^B Y$ and $W\succeq_H^B Z$ means $C_H(Y\cup W)=W$ and $C_H(Z\cup W)=W$ respectively Therefore, $C_H(C_H(Z\cup Y)\cup W)=C_H(Z\cup Y\cup W)=C_H(C_H(Z\cup W)\cup C_H(Y\cup W))=C_H(W)=W$. So, $W\succeq_H^B C_H(Y\cup Z)$ which implies $W\succeq_H^B D^O(C_H(Y\cup Z))$ according to lemma 5 (i).
iii) If we reverse the roles of the sides of the market, and develop again the proof of (i), we obtain $H^O(C_D(Y\cup Z))\succeq_D^B Y$ and $H^O(C_D(Y\cup Z))\succeq_D^B Z$. Consequently, $Y\succeq_H^B H^O(C_D(Y\cup Z))$ and $Z\succeq_H^B H^O(C_D(Y\cup Z))$ due to the counterposition of interests relative to the Blair´s partial orderings shown in theorem 3 .
iv) If we reverse the roles of the sides of the market, and develop again the proof of (ii), we obtain $W\succeq_D^B H^O(C_D(Y\cup Z))$. So, using again the counterposition of interests relative to the Blair´s partial orderings shown in theorem 3, we demonstrate that $Y\succeq_H^B W$ and $Z\succeq_H^B W$ imply $H^O(C_D(Y\cup Z))\succeq_H^B W$. $\square$

Lemmas 6 and 7 together prove the next theorem.

Theorem 5. *Let $(\mathbf{X},P)$ be a market of matching with contracts many-to-many where the preferences of all the agents satisfy substitutability. Then, $(S(\mathbf{X},P),\succeq_H^B,\vee_H,\wedge_H)$ is a lattice.*

Symmetrically, we can define the following operations between two any stable allocations Y and Z:

$$Y \vee_D Z = H^O(C_D\,(Y \cup Z))$$

and

$$Y \wedge_D Z = D^O(C_H\,(Y \cup Z)).$$

If we reverse the roles of the market-sides, and develop again this section, we achieve the next theorem:

THEOREM 6. *Let $(\mathbf{X}, P)$ be a market of matching with contracts many-to-many where the preferences of all the agents satisfy substitutability. Then, $(S(\mathbf{X}, P), \succeq_D^B, \vee_D, \wedge_D)$ is a lattice.*

In conclusion, the set $S(\mathbf{X}, P)$ is a lattice with respect to the Blair's partial ordering for hospitals and with respect to the Blair's partial ordering for doctors. Moreover, such lattices are dual according to the counterposition of interests shown in theorem 3.

6. Algorithm to compute the full set of stable allocations

In this section we introduce an algorithm that calculates the complete set of stable allocations for every market of contracts where the preferences of all the agents satisfy substitutability. For it, we use the results of the previous sections and follow the work of Martinez et al (2003).

Through this section, we will consider a fixed model of contracts $(\mathbf{X}, P)$ where P is a profile of substitutable preferences.

DEFINITION 13. *Given the original profile of preferences P and the contract $x \in \mathbf{X}$, the x-**truncation of P** is the profile P^x such that:*
1) All sets containing x are unacceptable to x_H according to $P^x_{x_H}$, this is

$$x \in S \Longrightarrow \varnothing \succ^x_{x_H} S.$$

2) P_{x_H} and $P^x_{x_H}$ coincide over all the sets that do not contain x, this is

$$x \notin S_1 \cup S_2 \text{ implies } S_1 \succ^x_{x_H} S_2 \Longleftrightarrow S_1 \succ_{x_H} S_2.$$

3) P_{x_H} and $P^x_{x_H}$ coincide over all the sets containing x, this is

$$x \in S_1 \cap S_2 \text{ implies } S_1 \succ^x_{x_H} S_2 \Longleftrightarrow S_1 \succ_{x_H} S_2.$$

4) All sets made unacceptable in $P^x_{x_H}$ are preferred to the original unacceptable sets, this is,
If $S_1, S_2 \subseteq \mathbf{X}$ are such that $x \in S_1$ and $S_1 \succ_{x_H} \varnothing \succ_{x_H} S_2$, then $S_1 \succ^x_{x_H} S_2$.
5) For every $i \in D \cup H \setminus \{x_H\}$ and $S_1, S_2 \subseteq \mathbf{X}_i$ we have $S_1 \succ^x_i S_2 \Longleftrightarrow S_1 \succ_i S_2$.

Conditions 3 and 4 are irrelevant for stability of allocations, but they guarantee the uniqueness of the x-truncation of P.

We have already shown that $D^O(\varnothing)$ and $H^O(\varnothing)$ are respectively the optimal for doctors and the optimal for hospitals stable allocations in the market $(\mathbf{X}, P)$. We use the original profile of preferences P to calculate them. If we use the x-truncation of P instead of P to run the algorithms, then we obtain for the market $(\mathbf{X}, P^x)$, the optimal for doctors stable allocation $D_x^O(\varnothing)$ and the optimal for hospitals stable allocation $H_x^O(\varnothing)$.

$C^x(.)$ and $R^x(.)$ will denote the choice set and the rejected set in the market $(\mathbf{X}, P^x)$; the set of all stable allocations in such market will be denoted $S(\mathbf{X}, P^x)$.

To calculate the set $S(\mathbf{X}, P)$ containing all the allocations which are stable under the original profile of preferences P, the following procedure must be iterated.
First, compute the optimal allocation $H^O(\varnothing)$ and $D^O(\varnothing)$.
Second, for each $x \in H^O(\varnothing) \setminus D^O(\varnothing)$, obtain x-truncation P^x and, using this new profile of preferences, compute the corresponding optimal for hospitals allocation $H_x^O(\varnothing)$.
Third, given $x \in H^O(\varnothing) \setminus D^O(\varnothing)$, It may occur that $H_x^O(\varnothing) \notin S(\mathbf{X}, P)$. Later we will see that, the stability of $H_x^O(\varnothing)$ with respect to the original profile P is ensured if $H_x^O(\varnothing)$ satisfies $C_{x_D}\left(H^O(\varnothing) \cup H_x^O(\varnothing)\right) = H_x^O(\varnothing)_{x_D}$. In such case, we keep $H_x^O(\varnothing)$ and start again from the beginning with P^x as input.
The algorithm stops when a new truncation is not possible because there is not a contract in the complement of $D^O(\varnothing)$ belonging to the optimal for hospitals allocation corresponding to the current truncated profile.
In the formal definition of the algorithm there is a dispensable step that improves the algorithm.

Algorithm to compute the complete set of stable allocations

Input

A market of contracts $(\mathbf{X}, P)$

Begin

Set $T^0(\mathbf{X}, P) := P$, $S^0(\mathbf{X}, P) := \left\{H^O(\varnothing)\right\}$and $k := 0$

Repeat

Step 1: Define

$\widetilde{T}(T^k(\mathbf{X}, P)) = \left\{P^{x_1 \ldots x_K x} : x \in H^O_{x_1 \ldots x_K}(\varnothing) \setminus D^O(\varnothing) \;\; \wedge \;\; P^{x_1 \ldots x_K} \in T^k(\mathbf{X}, P)\right\}$

Step 2: **If** $\widetilde{T}(T^k(\mathbf{X}, P)) = \varnothing$ set $T^{k+1}(\mathbf{X}, P) = \varnothing$ and $S^{k+1}(\mathbf{X}, P) = S^k(\mathbf{X}, P)$.

else for each truncation $P^{x_1 \ldots x_k x} \in \widetilde{T}(T^k(\mathbf{X}, P)) \neq \varnothing$ obtain the allocation $H^O_{x_1 \ldots x_K x}(\varnothing)$, which exists according to lemma 9

Step 3: Define

$T^*(T^k(\mathbf{X}, P)) = \left\{P^{x_1 \ldots x_k x} \in \widetilde{T}(T^k(\mathbf{X}, P)) : C_{x_D}\left(\left[H^O_{x_1 \ldots x_K x}(\varnothing)\right] \cup \left[H^O_{x_1 \ldots x_K}(\varnothing)\right]\right) = \left[H^O_{x_1 \ldots x_K x}(\varnothing)\right]_{x_D}\right\}$

Order the set $T^*(T^k(P))$ in an arbitrary way and let $\prec^{k+1}$ denote this ordering.

Step 4: Define

$\widehat{T}(T^k(\mathbf{X}, P)) = \{P^{x_1 \ldots x_k x} \in T^*(T^k(\mathbf{X}, P)) : \forall P^{x'_1 \ldots x'_k x'} \in T^*(T^k(\mathbf{X}, P))$
such that $P^{x_1 \ldots x_k x} \prec^{k+1} P^{x'_1 \ldots x'_k x'}, x' \in H^O_{x_1 \ldots x_K x}(\varnothing)\}$

Set $T^{k+1}(\mathbf{X}, P) := \widehat{T}(T^k(\mathbf{X}, P))$;
$S^{k+1}(\mathbf{X}, P) := S^k(\mathbf{X}, P) \cup \left\{H^O_{x_1 \ldots x_K x}(\varnothing) : P^{x_1 \ldots x_k x} \in T^{k+1}(\mathbf{X}, P)\right\}$
and $k := k + 1$

Until $T^k(\mathbf{X}, P) = \varnothing$.

End.

In the following example, we consider the market introduced in example 2 and calculate its corresponding full set of stable allocations. For a better exposition, some calculations will be realized in the appendix.

EXAMPLE 4. *Consider the market* $(\mathbf{X},P)$ *introduced in example 2. The doctors' optimal allocation and the hospitals' optimal allocation for this market are* $D^{O}(\varnothing)=\{x_{11},y_{21},y_{32}\}$ *and* $H^{O}(\varnothing)=\{x_{11},x_{21},x_{32}\}$, *respectively. They were calculated in examples 2 and 3.*

Input
the market $(\mathbf{X},P)$ *introduced in example 1.*
Begin
Set $T^{0}(\mathbf{X},P):=P$, $S^{0}(\mathbf{X},P):=\{H^{O}(\varnothing)\}$
Iterations
$\mathbf{i=1}$
Define
$\widetilde{T}(T^{0}(\mathbf{X},P))=\{P^{w}:w\in H^{O}(\varnothing)\setminus D^{O}(\varnothing)\ \wedge\ P\in T^{0}(\mathbf{X},P)\}=\{P^{x_{21}},P^{x_{32}}\}$
Since $\widetilde{T}(T^{k}(\mathbf{X},P))\neq\varnothing$, *obtain for each of its truncations the corresponding hospitals' optimal allocation:* $H^{O}_{x_{21,}}(\varnothing)=\{x_{11},y_{21},x_{32}\}$ *and* $H^{O}_{x_{32}}(\varnothing)=\{x_{11},x_{21},y_{32}\}$ *(see the calculations in the appendix).*
Define
$T^{*}(T^{0}(\mathbf{X},P))=\left\{P^{w}\in\widetilde{T}(T^{0}(\mathbf{X},P)):C_{x_D}\left(H^{O}_{w}(\varnothing)\cup H^{O}(\varnothing)\right)=\left[H^{O}_{w}(\varnothing)\right]_{x_D}\right\}=$ $\{P^{x_{21}},P^{x_{32}}\}$
in fact, $C_{d_2}\left(H^{O}_{x_{21}}(\varnothing)\cup H^{O}(\varnothing)\right)=C_{d_2}(\{y_{21}\}\cup\{x_{21}\})=\{y_{21}\}=\left[H^{O}_{x_{21}}(\varnothing)\right]_{d_2}$
and $C_{d_3}\left(H^{O}_{x_{32}}(\varnothing)\cup H^{O}(\varnothing)\right)=C_{d_3}(\{y_{32}\}\cup\{x_{32}\})=\{y_{32}\}=\left[H^{O}_{x_{32}}(\varnothing)\right]_{d_3}$.
Considering the ordering $P^{x_{21}}\prec^{1}P^{x_{32}}$ *over the profiles in* $T^{*}(T^{0}(\mathbf{X},P))$, *define*
$\widehat{T}(T^{0}(\mathbf{X},P))=\{P^{w}\in T^{*}(T^{0}(\mathbf{X},P)):\forall P^{w'}\in T^{*}(T^{0}(\mathbf{X},P))$ *such that* $P^{w}\prec^{1}P^{w'},w'\in H^{O}_{w}(\varnothing)\}=\{P^{x_{21}},P^{x_{32}}\}$
in fact, $x_{32}\in H^{O}_{x_{21,}}(\varnothing)$ *which implies* $P^{x_{21}}\in\widehat{T}(T^{0}(\mathbf{X},P))$, *and* $P^{x_{32}}\in\widehat{T}(T^{0}(\mathbf{X},P))$ *due to the definition of* $\prec^{1}$.
Then, set $T^{1}(\mathbf{X},P):=\widehat{T}(T^{0}(\mathbf{X},P))=\{P^{x_{21}},P^{x_{32}}\}$,
$S^{1}(\mathbf{X},P):=S^{0}(\mathbf{X},P)\cup\{H^{O}_{w}(\varnothing):P^{w}\in T^{1}(\mathbf{X},P)\}$
So $S^{1}(\mathbf{X},P)=\{H^{O}(\varnothing),H^{O}_{x_{21}}(\varnothing),H^{O}_{x_{32}}(\varnothing)\}$. *This finishes the first stage.*
Since $T^{1}(\mathbf{X},P)\neq\varnothing$, *realize a new iteration.*
$\mathbf{i=2}$
Define
$\widetilde{T}(T^{1}(\mathbf{X},P))=\{P^{wz}:z\in H^{O}_{w}(\varnothing)\setminus D^{O}(\varnothing)\ \wedge\ P^{w}\in T^{1}(\mathbf{X},P)\}=\{P^{x_{21}x_{32}},P^{x_{32}x_{21}}\}$
Since $\widetilde{T}(T^{1}(\mathbf{X},P))\neq\varnothing$, *obtain for each of its truncations the corresponding hospitals' optimal allocation:*
$H^{O}_{x_{21}x_{32}}(\varnothing)=H^{O}_{x_{32}x_{21}}(\varnothing)=\{x_{11},y_{21},y_{32}\}$ *(see the calculations in the appendix).*
Define
$T^{*}(T^{1}(\mathbf{X},P))=\left\{P^{wz}\in\widetilde{T}(T^{1}(\mathbf{X},P)):C_{x_D}\left(H^{O}_{wz}(\varnothing)\cup H^{O}(\varnothing)\right)=\left[H^{O}_{wz}(\varnothing)\right]_{x_D}\right\}=$ $\{P^{x_{21}x_{32}},P^{x_{32}x_{21}}\}$
In fact, $C_{d_3}\left(H^{O}_{x_{21}x_{32}}(\varnothing)\cup H^{O}(\varnothing)\right)=C_{d_3}(\{y_{32}\}\cup\{x_{32}\})=\{y_{32}\}=\left[H^{O}_{x_{21}x_{32}}(\varnothing)\right]_{d_3}$
and $C_{d_2}\left(H^{O}_{x_{32}x_{21}}(\varnothing)\cup H^{O}(\varnothing)\right)=C_{d_2}(\{y_{21}\}\cup\{x_{21}\})=\{y_{21}\}=\left[H^{O}_{x_{32}x_{21}}(\varnothing)\right]_{d_2}$.
Considering the ordering $P^{x_{21}x_{32}}\prec^{2}P^{x_{32}x_{21}}$ *over the profiles in* $T^{*}(T^{1}(\mathbf{X},P))$, *define*
$\widehat{T}(T^{1}(\mathbf{X},P))=\{P^{wz}\in T^{*}(T^{0}(\mathbf{X},P)):\forall P^{w'z'}\in T^{*}(T^{0}(\mathbf{X},P))$ *such that* $P^{wz}\prec^{2}P^{w'z'},z'\in H^{O}_{wz}(\varnothing)\}=\{P^{x_{32}x_{21}}\}$
in fact, $P^{x_{21}x_{32}}\notin\widehat{T}(T^{1}(\mathbf{X},P))$ *because* $x_{32}\notin H^{O}_{x_{32}x_{21}}(\varnothing)$, *and* $P^{x_{32}x_{21}}\in\widehat{T}(T^{1}(\mathbf{X},P))$

due to the definition of $\prec^2$.
Then, set $T^2(\mathbf{X},P) := \widehat{T}(T^1(\mathbf{X},P)) = \{P^{x_{32}x_{21}}\}$
$S^2(\mathbf{X},P) := S^1(\mathbf{X},P) \cup \left\{H^O_{wz}(\varnothing) : P^{wz} \in T^2(\mathbf{X},P)\right\}$
So $S^2(\mathbf{X},P) = \left\{H^O(\varnothing), H^O_{x_{21}}(\varnothing), H^O_{x_{32}}(\varnothing), H^O_{x_{32}x_{21}}(\varnothing)\right\}$ *This finishes the first stage.*
Since $T^2(\mathbf{X},P) \neq \varnothing$, *realize a new iteration.*
$\mathbf{i=3}$
Define
$\widetilde{T}(T^2(\mathbf{X},P)) = \left\{P^{wzu} : u \in H^O_{wz}(\varnothing) \setminus D^O(\varnothing) \ \wedge \ P^{wz} \in T^2(\mathbf{X},P)\right\} = \varnothing$
Since $\widetilde{T}(T^2(\mathbf{X},P)) = \varnothing$, *set* $T^3(\mathbf{X},P) = \varnothing$ *and*
$S^3(\mathbf{X},P) = S^2(\mathbf{X},P) = H^O(\varnothing), H^O_{x_{21}}(\varnothing), H^O_{x_{32}}(\varnothing), H^O_{x_{32}x_{21}}(\varnothing)$
End.

Therefore, the complete set of stable allocations for the market $(\mathbf{X},P)$ *is:*
$S(\mathbf{X},P) = S^3(\mathbf{X},P) = \left\{H^O(\varnothing), H^O_{x_{21}}(\varnothing), H^O_{x_{32}}(\varnothing), H^O_{x_{32}x_{21}}(\varnothing)\right\} =$
$\{\{x_{11},x_{21},x_{32}\}, \ \{x_{11},y_{21},x_{32}\}, \ \{x_{11},x_{21},y_{32}\}, \ \{x_{11},y_{21},y_{32}\}\}.$ □

Hereinafter, we will prove that our algorithm works.

THEOREM 7. *Let K be the stage where the algorithm to compute the complete set of stable allocations stops, i.e., $T^K(\mathbf{X},P) = \varnothing$. If the preferences of all the agents satisfy substitutability in the original profile of preferences P, then $S^K(\mathbf{X},P) = S(\mathbf{X},P)$.*

Before demonstrating the previous theorem, it is necessary to prove some lemmas.

LEMMA 8. *Given $A \subseteq \mathbf{X}$, $C^x_{x_H}(A) = C_{x_H}(A \setminus \{x\})$.*

PROOF. Since all subsets of A containing x are unacceptable for x_H according to the preferences $P^x_{x_H}$, $C^x_{x_H}(A)$ is the subset of $A \setminus \{x\}$ in the top of the list of preferences $P^x_{x_H}$. Such set is $C_{x_H}(A \setminus \{x\})$ according to the definition of x-truncation. □

LEMMA 9. *Given a contract $x \in \mathbf{X}$, suppose that the original preferences of x_H, P_{x_H}, satisfy substitutability. Then, the modified preferences $P^x_{x_H}$ also satisfy substitutability.*

PROOF. Given $X \subseteq Y \subseteq \mathbf{X}$, we will show that $R^x_{x_H}(X) \subseteq R^x_{x_H}(Y)$.
Suppose that $z \notin R^x_{x_H}(Y)$, then, three cases are possible:
i) $z \notin Y_{x_H}$, so $z \notin X_{x_H}$ and consequently $z \notin R^x_{x_H}(X)$.
ii) $z \in C^x_{x_H}(Y) \cap X$, then $z \in C_{x_H}(Y \setminus \{x\})$ according to lemma 8. This implies $z \in C_{x_H}(X \setminus \{x\})$ due to substitutability. Thus, $z \in C^x_{x_H}(X)$ because of lemma 8. Therefore, $z \notin R^x_{x_H}(X)$.
iii) $z \in C^x_{x_H}(Y) \setminus X$, then $z \notin R^x_{x_H}(X)$. □

Given a contract $x \in X$, let $S^x(\mathbf{X},P)$ denote the set of allocations that are stable under the profile of preferences P^x and satisfy the following condition

$$S^x(\mathbf{X},P) = \left\{X \in S(\mathbf{X},P^x) / C_{x_D}\left(X \cup H^O(\varnothing)\right) = X_{x_D}\right\}$$

In lemma 10 we will prove that $S^x(\mathbf{X},P) \subseteq S(\mathbf{X},P)$ for all $x \in H^O(\varnothing) \setminus D^O(\varnothing)$. So, the condition $C_{x_D}\left(X \cup H^O(\varnothing)\right) = X_{x_D}$ is enough to ensure the stability under the original profile of preferences P of the allocation X which is stable with respect to P^x.

LEMMA 10. *Given $x \in H^O(\varnothing) \setminus D^O(\varnothing)$. If $X \in S^x(\mathbf{X},P)$, then $X \in S(\mathbf{X},P)$.*

PROOF. First, $x \notin X$ because X is an individually rational allocation under the profile P^x according to the hypothesis, and every set of contracts containing x is unacceptable to x_H under the profile P^x. As a consequence, $C_i(X) = C_i^x(X) = X_i$ for every $i \in D \cup H$. So, X is an individually rational allocation under the original profile of preferences P.
Second, assume the existence of a contract $y \in \mathbf{X} \setminus X$ such that $y \in C_{y_H}(X \cup \{y\}) \cap C_{y_D}(X \cup \{y\})$.
Observe that $y \neq x$. In fact, if $y = x$ then $y \in H^O(\varnothing) \setminus X$ and $C_{y_D}\left(X \cup H^O(\varnothing)\right) = X_{y_D}$ (due to the hypothesis $x \in H^O(\varnothing)$ and $X \in S^x(\mathbf{X},P)$ respectively); consequently, $y \notin C_{y_D}(X \cup \{y\})$ because of substitutability, this is a contradiction.
Therefore, $x \notin X \cup \{y\}$ and, as a consequence, $C_i(X \cup \{y\}) = C_i^x(X \cup \{y\})$ for all $i \in D \cup H$.
So, the assumption about the existence of a contract $y \in \mathbf{X} \setminus X$. such that $y \in C_{y_H}(X \cup \{y\}) \cap C_{y_D}(X \cup \{y\})$ implies the existence of a contract $y \in \mathbf{X} \setminus X$ such that $y \in C_{y_H}^x(X \cup \{y\}) \cap C_{y_D}^x(X \cup \{y\})$ contradicting the hypothesis $X \in S(\mathbf{X},P^x)$. □

LEMMA 11. *Let P^x be a x-truncation such that $C_{x_D}\left(H^O(\varnothing) \cup H_x^O(\varnothing)\right) = \left[H_x^O(\varnothing)\right]_{x_D}$ and $x \in H^O(\varnothing) \setminus D^O(\varnothing)$. Then, $X \in S(\mathbf{X},P^x)$ implies $X \in S(\mathbf{X},P)$.*

PROOF. $X \in S(\mathbf{X},P^x)$ implies $C_{x_D}^x\left(X \cup H_x^O(\varnothing)\right) = X_{x_D}$. In fact, P^x is a profile of substitutable preferences, so $S(\mathbf{X},P^x)$ is a lattice with respect to the Blair's partial ordering for doctors as we proved in the previous section, and $H_x^O(\varnothing)$ is the worst stable allocation according to such partial ordering. Since x_D has the same preferences under both profiles P^x and P, we have $C_{x_D}\left(X \cup H_x^O(\varnothing)\right) = X_{x_D}$. Because of the hypothesis, $C_{x_D}\left(H^O(\varnothing) \cup H_x^O(\varnothing)\right) = \left[H_x^O(\varnothing)\right]_{x_D}$. Then,
$C_{x_D}\left(X \cup H^O(\varnothing)\right) = X_{x_D}$ due to the transitivity of the partial ordering. So, $X \in S^x(\mathbf{X},P)$ which implies $X \in S(\mathbf{X},P)$ according to lemma 10.. □

LEMMA 12. *Let X be an allocation such that $X \in S(\mathbf{X},P)$ and $C_i^x(X) = X_i$ for all $i \in D \cup H$. Then, $X \in S(\mathbf{X},P^x)$.*

PROOF. Observe that $x \notin X$ because X is an individually rational allocation under the profile of preferences P^x, and no set of contracts containing x is acceptable for x_H under such profile.
Suppose the existence of a contract $y \in \mathbf{X} \setminus X$ such that $y \in C_{y_H}^x(X \cup \{y\}) \cap C_{y_D}^x(X \cup \{y\})$. Observe that $y \in C_{y_H}^x(X \cup \{y\})$ implies $x \neq y$.
So, $x \notin X \cup \{y\}$. As a consequence, for every $i \in D \cup H$ we have $C_i^x(X \cup \{y\}) = C_i(X \cup \{y\})$.
Then, the contract $y \in \mathbf{X} \setminus X$ also satisfies $y \in C_{y_H}(X \cup \{y\}) \cap C_{y_D}(X \cup \{y\})$ and the hypothesis $X \in S(\mathbf{X},P)$ is contradicted. □

Corollary 2. *Let P^x and P^y be two truncations of the profile P such that $H_x^O(\varnothing) \in S(\mathbf{X}, P)$ and $H_y^O(\varnothing) \in S(\mathbf{X}, P)$. If $x \notin H_y^O(\varnothing)$, then $H_y^O(\varnothing) \in S(\mathbf{X}, P^x)$.*

Proof. $H_y^O(\varnothing) \in S(\mathbf{X}, P)$ implies $H_y^O(\varnothing)_i = C_i\left(H_y^O(\varnothing)\right)$ for every $i \in D \cup H$. Since $x \notin H_y^O(\varnothing)$, it follows that $H_y^O(\varnothing)_i = C_i^x\left(H_y^O(\varnothing)\right)$ $\forall i \in D \cup H$. Then, $H_y^O(\varnothing) \in S(\mathbf{X}, P^x)$ according to lemma 12. □

Lemma 13. *If $x \in H^O(\varnothing) \cap D^O(\varnothing)$, then $x \in X$ for every $X \in S(\mathbf{X}, P)$.*

Proof. Assume $x \in H^O(\varnothing) \cap D^O(\varnothing)$ but $x \notin X'$ for some $X' \in S(\mathbf{X}, P)$. Since $S(\mathbf{X}, P)$ is a lattice with respect to the Blair's partial ordering for hospitals and with respect to the Blair's partial ordering for doctors, as we proved in the previous section, we have $C_{x_H}\left(H^O(\varnothing) \cup X'\right) = \left[H^O(\varnothing)\right]_{x_H}$ and $C_{x_D}\left(D^O(\varnothing) \cup X'\right) = \left[D^O(\varnothing)\right]_{x_D}$. As a consequence, $x \in C_{x_H}\left(H^O(\varnothing) \cup X'\right) \cap C_{x_D}\left(D^O(\varnothing) \cup X'\right)$. Then, $x \in C_{x_H}(\{x\} \cup X') \cap C_{x_D}(\{x\} \cup X')$ since x_H and x_D have substitutable preferences. So, $X' \in S(\mathbf{X}, P)$ is contradicted. □

Lemma 14. *If $X \in S(\mathbf{X}, P)$ is an allocation such that $X \neq H^O(\varnothing)$, then $X \in S(\mathbf{X}, P^x)$ for some profile P^x with $x \in H^O(\varnothing) \setminus D^O(\varnothing)$ and $x \notin X$.*

Proof. $X \neq H^O(\varnothing)$ implies the existence of a contract $x \in H^O(\varnothing) \setminus X$. In fact, in other case, $H^O(\varnothing) \subseteq X$ together with the lattice structure of $S(\mathbf{X}, P)$ with respect to the Blair's partial ordering for hospitals would imply $H^O(\varnothing) = C_H\left(H^O(\varnothing) \cup X\right) = C_H(X) = X$ which is a contradiction.
Since $x \notin X$, lemma 13 implies $x \notin D^O(\varnothing)$. Consider the profile P^x. Observe that $C_i^x(X) = X_i$ for all $i \in D \cup H$ because $C_i(X) = X_i$ for every $i \in D \cup H$ and $x \notin X$. Then, $X \in S(\mathbf{X}, P^x)$ according to lemma 12. □

<u>Remark:</u> *Let P^x be a truncated profile of preferences such that $x \in H^O(\varnothing) \setminus D^O(\varnothing)$* and $H_x^O(\varnothing)$ *satisfies* $C_{x_D}\left(H^O(\varnothing) \cup H_x^O(\varnothing)\right) = \left[H_x^O(\varnothing)\right]_{x_D}$. *Then,* $S(\mathbf{X}, P) \supseteq S(\mathbf{X}, P^x)$ according to lemma 11 and $H^O(\varnothing) \notin S(\mathbf{X}, P^x)$ *because no set of contracts containing* x is acceptable for x_H *under* P^x. *Therefore,* $\mid S(\mathbf{X}, P) \mid > \mid S(\mathbf{X}, P^x) \mid$.

Corollary 3. *If $X \in S(\mathbf{X}, P)$ is an allocation such that $X \neq H^O(\varnothing)$, then a succession of contracts $x_1, ..., x_n$ such that $X = H^O_{x_1 \cdots x_n}(\varnothing) \in S(\mathbf{X}, P^{x_1 \cdots x_n})$ exists.*

Proof. By hypothesis, $X \in S(\mathbf{X}, P)$and $X \neq H^O(\varnothing)$. Then, $X \in S(\mathbf{X}, P^x)$ for some profile P^x with $x \in H^O(\varnothing) \setminus D^O(\varnothing)$ and $x \notin X$ according to lemma 14. If $X = H_x^O(\varnothing)$, the proof ends. Else, because of the above remark, $\mid S(\mathbf{X}, P) \mid > \mid S(\mathbf{X}, P^x) \mid$; then, we apply again lemma 14 replacing P and X with P^x and $H_x^O(\varnothing)$ respectively. Since $\mid S(\mathbf{X}, P) \mid < \infty$, the statement of corollary 3 follows. □

Proof of theorem 7 : First, lemma 11 implies $S^1(\mathbf{X},P) \subseteq S(\mathbf{X},P)$. Applying iteratively lemma 11to successive stages , we have $S^K(\mathbf{X},P) \subseteq S(\mathbf{X},P)$.
Second, suppose that $X \in S(\mathbf{X},P)$. Then, there exists $k \leq K$ such that $X \in S^k(\mathbf{X},P)$ according to corollary 3. Consequently, $S(\mathbf{X},P) \subseteq S^K(\mathbf{X},P)$.
Therefore, $S(\mathbf{X},P) = S^K(\mathbf{X},P)$. ■

7. Appendix

I) Calculation of the hospitals' optimal allocation under the truncated profile $P^{x_{21}}$ using the HOA.

$P^{x_{21}}_{d_1} : \{x_{11}, x_{12}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{x_{12}\} \succ_{h_1} \varnothing$
$P^{x_{21}}_{d_2} : \{y_{21}\} \succ_{d_2} \{x_{21}\} \succ_{d_2} \varnothing$
$P^{x_{21}}_{d_3} : \{y_{32}\} \succ_{d_3} \{x_{32}\} \succ_{d_3} \varnothing$
$P^{x_{21}}_{h_1} : \{x_{11}, y_{21}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{y_{21}\} \succ_{h_1} \varnothing$
$P^{x_{21}}_{h_2} : \{x_{32}\} \succ_{h_2} \{y_{32}\} \succ_{h_2} \varnothing$

Input
$\mathbf{X} = \{x_{11}, x_{12}, x_{21}, y_{21}, x_{32}, y_{32}\}$; the profile of substitutable preferences $P^{x_{21}}$; the IRD allocation $\varnothing$.
Begin:
$Y^0 = \varnothing$
$\mathbf{i = 1}$
Calculate

$$I(H, Y^0) = \mathbf{X} \text{ in fact:}$$

$x \in C_{x_D}(\{x\} \cup \varnothing)$ for all $x \in \mathbf{X}$.
Obtain

$$C_H\left(I(H, Y^0)\right) = \{x_{11}, y_{21}, x_{32}\} \text{ in fact:}$$

$C_{h_1}(I(h_1, Y^0)) = C_{h_1}(\{x_{11}, x_{21}, y_{21}\}) = \{x_{11}, y_{21}\}$; $C_{h_2}(I(h_2, Y^0)) = C_{h_2}(\{x_{12}, x_{32}, y_{32}\}) = \{x_{32}\}$
Define

$$Y^1 = C_D(C_H\left(I(H, Y^0)\right)) = \{x_{11}, y_{21}, x_{32}\} \text{ in fact:}$$

$C_{d_1}(C_H\left(I(H, Y^0)\right)) = C_{d_1}(\{x_{11}\}) = \{x_{11}\}$; $C_{d_2}(C_H\left(I(H, Y^0)\right)) = C_{d_2}(\{y_{21}\}) = \{y_{21}\}$; $C_{d_3}(C_H\left(I(H, Y^0)\right)) = C_{d_3}(\{x_{32}\}) = \{x_{32}\}$
Since $Y^1 \neq Y^0$, realize a new iteration.
$\mathbf{i = 2}$
Calculate

$$I(H, Y^1) = \{x_{11}, x_{12}, y_{21}, x_{32}, y_{32}\} \text{ in fact:}$$

$x_{11} \in C_{d_1}\left(\{x_{11}\} \cup Y^1\right) = \{x_{11}\}$; $x_{12} \in C_{d_1}\left(\{x_{12}\} \cup Y^1\right) = \{x_{11}, x_{12}\}$; $x_{21} \notin C_{d_2}\left(\{x_{21}\} \cup Y^1\right) = \{y_{21}\}$; $y_{21} \in C_{d_2}\left(\{y_{21}\} \cup Y^1\right) = \{y_{21}\}$; $x_{32} \in C_{d_3}\left(\{x_{32}\} \cup Y^1\right) = \{x_{32}\}$; $y_{32} \in C_{d_3}\left(\{y_{32}\} \cup Y^1\right) = \{y_{32}\}$.
Obtain

$$C_H\left(I(H, Y^1)\right) = \{x_{11}, y_{21}, x_{32}\} \text{ in fact:}$$

$C_{h_1}(I(h_1,Y^1)) = C_{h_1}(\{x_{11},y_{21}\}) = \{x_{11},y_{21}\}$
$C_{h_2}(I(h_2,Y^1)) = C_{h_2}(\{x_{12},x_{32},y_{32}\}) = \{x_{32}\}$
Define

$$Y^2 = C_D(C_H\left(I(H,Y^1)\right)) = \{x_{11},y_{21},x_{32}\} \text{ in fact:}$$

$C_{d_1}(C_H\left(I(H,Y^1)\right)) = C_{d_1}(\{x_{11}\}) = \{x_{11}\}\,;\ C_{d_2}(C_H\left(I(H,Y^1)\right)) = C_{d_2}(\{y_{21}\}) = \{y_{21}\}\,;\ C_{d_3}(C_H\left(I(H,Y^1)\right)) = C_{d_3}(\{x_{32}\}) = \{x_{32}\}$
Since $Y^2 = Y^1$ the algorithm stops.
Output
$\{x_{11},y_{21},x_{32}\}$

II) Calculation of the hospitals' optimal allocation under the truncated profile $P^{x_{32}}$ using the HOA.

$P^{x_{32}}_{d_1} : \{x_{11},x_{12}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{x_{12}\} \succ_{h_1} \varnothing$
$P^{x_{32}}_{d_2} : \{y_{21}\} \succ_{d_2} \{x_{21}\} \succ_{d_2} \varnothing$
$P^{x_{32}}_{d_3} : \{y_{32}\} \succ_{d_3} \{x_{32}\} \succ_{d_3} \varnothing$
$P^{x_{32}}_{h_1} : \{x_{11},x_{21}\} \succ_{h_1} \{x_{11},y_{21}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{x_{21}\} \succ_{h_1} \{y_{21}\} \succ_{h_1} \varnothing$
$P^{x_{32}}_{h_2} : \{y_{32}\} \succ_{h_2} \varnothing$

Input
$\mathbf{X} = \{x_{11},x_{12},x_{21},y_{21},x_{32},y_{32}\}$; the profile of substitutable preferences $P^{x_{32}}$; the IRD allocation $\varnothing$.
Begin:
$Y^0 = \varnothing$
$\mathbf{i = 1}$
Calculate

$$I(H,Y^0) = \mathbf{X} \text{ in fact:}$$

$x \in C_{x_D}(\{x\} \cup \varnothing)$ for every $x \in \mathbf{X}$.
Obtain

$$C_H\left(I(H,Y^0)\right) = \{x_{11},x_{21},y_{32}\} \text{ in fact:}$$

$C_{h_1}(I(h_1,Y^0)) = C_{h_1}(\{x_{11},x_{21},y_{21}\}) = \{x_{11},x_{21}\}\,; C_{h_2}(I(h_2,Y^0)) = C_{h_2}(\{x_{12},x_{32},y_{32}\}) = \{y_{32}\}$
Define

$$Y^1 = C_D(C_H\left(I(H,Y^0)\right)) = \{x_{11},x_{21},y_{32}\} \text{ in fact:}$$

$C_{d_1}(C_H\left(I(H,Y^0)\right)) = C_{d_1}(\{x_{11}\}) = \{x_{11}\}\,;\ C_{d_2}(C_H\left(I(H,Y^0)\right)) = C_{d_2}(\{x_{21}\}) = \{x_{21}\}\,;\ C_{d_3}(C_H\left(I(H,Y^0)\right)) = C_{d_3}(\{y_{32}\}) = \{y_{32}\}$
Since $Y^1 \neq Y^0$ realize a new iteration
$\mathbf{i = 2}$
Calculate

$$I(H,Y^1) = \{x_{11},x_{12},x_{21},y_{21},y_{32}\} \text{ in fact:}$$

$x_{11} \in C_{d_1}\left(\{x_{11}\} \cup Y^1\right) = \{x_{11}\}\,;\ x_{12} \in C_{d_1}\left(\{x_{12}\} \cup Y^1\right) = \{x_{11},x_{12}\}\,;\ x_{21} \in C_{d_2}\left(\{x_{21}\} \cup Y^1\right) = \{x_{21}\}\,; y_{21} \in C_{d_2}\left(\{y_{21}\} \cup Y^1\right) = \{y_{21}\}\,; x_{32} \notin C_{d_3}\left(\{x_{32}\} \cup Y^1\right) =$

$\{y_{32}\}$; $y_{32} \in C_{d_3}\left(\{y_{32}\} \cup Y^1\right) = \{y_{32}\}$.
Obtain

$$C_H\left(I(H, Y^1)\right) = \{x_{11}, x_{21}, y_{32}\} \text{ in fact:}$$

$C_{h_1}(I(h_1, Y^1)) = C_{h_1}(\{x_{11}, x_{21}, y_{21}\}) = \{x_{11}, x_{21}\}$; $C_{h_2}(I(h_2, Y^1)) = C_{h_2}(\{x_{12}, y_{32}\}) = \{y_{32}\}$
Define

$$Y^2 = C_D(C_H\left(I(H, Y^1)\right)) = \{x_{11}, x_{21}, y_{32}\} \text{ in fact:}$$

$C_{d_1}(C_H\left(I(H, Y^1)\right)) = C_{d_1}(\{x_{11}\}) = \{x_{11}\}$; $C_{d_2}(C_H\left(I(H, Y^1)\right)) = C_{d_2}(\{y_{21}\}) = \{x_{21}\}$; $C_{d_3}(C_H\left(I(H, Y^1)\right)) = C_{d_3}(\{x_{32}\}) = \{x_{32}\}$
Since $Y^2 = Y^1$ the algorithm stops.
Output
$\{x_{11}, x_{21}, y_{32}\}$

III) Calculation of the hospitals' optimal allocation under the truncated profile $P^{x_{21}x_{32}} = P^{x_{32}x_{21}}$ using the HOA.

$P^{x_{21}x_{32}}_{d_1} : \{x_{11}, x_{12}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{x_{12}\} \succ_{h_1} \varnothing$
$P^{x_{21}x_{32}}_{d_2} : \{y_{21}\} \succ_{d_2} \{x_{21}\} \succ_{d_2} \varnothing$
$P^{x_{21}x_{32}}_{d_3} : \{y_{32}\} \succ_{d_3} \{x_{32}\} \succ_{d_3} \varnothing$
$P^{x_{21}x_{32}}_{h_1} : \{x_{11}, y_{21}\} \succ_{h_1} \{x_{11}\} \succ_{h_1} \{y_{21}\} \succ_{h_1} \varnothing$
$P^{x_{21}x_{32}}_{h_2} : \{y_{32}\} \succ_{h_2} \varnothing$

Input
$\mathbf{X} = \{x_{11}, x_{12}, x_{21}, y_{21}, x_{32}, y_{32}\}$; the profile of substitutable preferences $P^{x_{21}x_{32}} = P^{x_{32}x_{21}}$; the IRD allocation $\varnothing$.
Start:
$Y^0 = \varnothing$
$\mathbf{i = 1}$
Calculate

$$I(H, Y^0) = \mathbf{X} \text{ in fact:}$$

$x \in C_{x_D}(\{x\} \cup \varnothing)$ for all $x \in \mathbf{X}$.
Obtain

$$C_H\left(I(H, Y^0)\right) = \{x_{11}, y_{21}, y_{32}\} \text{ in fact:}$$

$C_{h_1}(I(h_1, Y^0)) = C_{h_1}(\{x_{11}, x_{21}, y_{21}\}) = \{x_{11}, y_{21}\}$; $C_{h_2}(I(h_2, Y^0)) = C_{h_2}(\{x_{12}, x_{32}, y_{32}\}) = \{y_{32}\}$
Define

$$Y^1 = C_D(C_H\left(I(H, Y^0)\right)) = \{x_{11}, y_{21}, y_{32}\} \text{ in fact:}$$

$C_{d_1}(C_H\left(I(H, Y^0)\right)) = C_{d_1}(\{x_{11}\}) = \{x_{11}\}$; $C_{d_2}(C_H\left(I(H, Y^0)\right)) = C_{d_2}(\{y_{21}\}) = \{y_{21}\}$; $C_{d_3}(C_H\left(I(H, Y^0)\right)) = C_{d_3}(\{y_{32}\}) = \{y_{32}\}$
Since $Y^1 \neq Y^0$ realize a new iteration.
$\mathbf{i = 2}$
Calculate

$$I(H, Y^1) = \{x_{11}, x_{12}, y_{21}, y_{32}\} \text{ in fact:}$$

$x_{11} \in C_{d_1}\left(\{x_{11}\} \cup Y^1\right) = \{x_{11}\}$; $x_{12} \in C_{d_1}\left(\{x_{12}\} \cup Y^1\right) = \{x_{11}, x_{12}\}$; $x_{21} \notin$

$C_{d_2}\left(\{x_{21}\}\cup Y^1\right)=\{y_{21}\}$; $y_{21}\in C_{d_2}\left(\{y_{21}\}\cup Y^1\right)=\{y_{21}\}$; $x_{32}\notin C_{d_3}\left(\{x_{32}\}\cup Y^1\right)=\{y_{32}\}$; $y_{32}\in C_{d_3}\left(\{y_{32}\}\cup Y^1\right)=\{y_{32}\}$.
Obtain

$$C_H\left(I(H,Y^1)\right)=\{x_{11},y_{21},y_{32}\}\text{ in fact:}$$

$C_{h_1}(I(h_1,Y^1))=C_{h_1}(\{x_{11},y_{21}\})=\{x_{11},y_{21}\}$; $C_{h_2}(I(h_2,Y^1))=C_{h_2}(\{x_{12},y_{32}\})=\{y_{32}\}$
Define

$$Y^2=C_D(C_H\left(I(H,Y^1)\right))=\{x_{11},y_{21},y_{32}\}\text{ in fact:}$$

$C_{d_1}(C_H\left(I(H,Y^1)\right))=C_{d_1}(\{x_{11}\})=\{x_{11}\}$; $C_{d_2}(C_H\left(I(H,Y^1)\right))=C_{d_2}(\{y_{21}\})=\{x_{21}\}$; $C_{d_3}(C_H\left(I(H,Y^1)\right))=C_{d_3}(\{x_{32}\})=\{x_{32}\}$
Since $Y^2=Y^1$ the algorithm stops.
Output
$\{x_{11},y_{21},y_{32}\}$

8. Bibliography

[1] Blair C. (1988) The lattice Structure of the Set of Stable Matchings with Multiple Partners. Mathematics of Operations Research; **13(4)**, 619-628.

[2]Gale and Shapley (1962) College Admisions and the Stability of Marriage. American.Math Monthly, **69**, 9-15.

[3] Hatfield J.and Milgrom P. (2005) Matching with contracts. The American Economic Review, **95(4)**, 913-935.

[4] Hatfiel W.and Kojima F. (2010) Substitutes and Stability for Matching with Contracts. Journal of Economic Theory, **145**, 1704-1723.

[5] Hatfield J.and Kominers S. Contract Design and Stability in Many to many Matching. (working paper)

[6] Klaus B.and Walzl M. (2009) Stable Many to Many Matching with Contracts. Journal of Mathematical Economics, **45**, 422- 434

[7] Knuth D. (1976) Marriages Stables. Montréal: Les Presses de l'Universite de Montréal.

[8] Martinez R., Massó J., Neme A.and Oviedo J. (2001) On the Lattice Structure of the Set of Stable Matchings for a Many to One Model. Optimization, **50**, 439-457.

[9] Martinez R., Massó J., Neme A., Oviedo J. (2004) An Algorithm to Compute the Full Set of Many to Many Stable Matching. Mathematical Social Sciences, **47(2)**, pg 187-210.

[10] McVitie D. and Wilson L. (1970). Stable Marriage Assignment for Unequal Sets. BIT, **10**, 295-309.

[11] McVitie D. and Wilson L. (1971). The Stable Marriage Problem. Comunications of the ACM, **14**, 486-496.

[12] Roth A.(1984) Stability and Polarization of Interests in Job Matching. Econométrica, **52**, 47-57.

[13] Roth A. (1985) Conflict and Coincidence: Some New Result and Open Questions of Interest in Job Matching.

[14] Roth A. (1985) Conflict and Coincidence of Interest in Job Matching. Math. Operational Research, **10**, 379-389.

[15] Roth A. (2008). Deferred Acceptance Algorithms: History, Ttheory, Practice and Open Questions.
International Journal of Game Theory, **36**, 537-569.

[16] Roth A.and Sotomayor M. Book Two Sided Matching, a Study in Game Theoretic Modeling and Analysis Cambridge University Press