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RESEARCH ARTICLE

# Second-best efficiency of allocation rules: strategy-proofness and single-peaked preferences with multiple commodities

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**Abstract** We study strategy-proof allocation rules in economies with perfectly divisible multiple commodities and single-peaked preferences. In this setup, it is known that the incompatibility among *strategy-proofness, Pareto efficiency* and *non-dictatorship* arises in contrast with the Sprumont (Econometrica 59:509–519, 1991) one commodity model. We first investigate the existence problem of strategy-proof and second-best efficient rules, where a strategy-proof rule is second-best efficient if it is not Pareto-dominated by any other strategy-proof rules. We show that there exists an egalitarian rational (consequently, non-dictatorial) strategy-proof rule satisfying second-best efficiency. Second, we give a new characterization of *the generalized uniform rule* with the second-best efficiency in two-agent case.

**Keywords** Strategy-proofness · Single-peaked preference · Second-best efficiency · Generalized uniform rule

**JEL Classification** D63 · D71 · D78

## 1 Introduction

Ever since Sprumont (1991), resource allocation problems in economies with single-peaked preferences have been studied by many authors. As is pointed out in Sprumont

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(1991), a single-peaked preference model may have several important interpretations. One possible interpretation is the "fixed-price economy" interpretation. In this interpretation, the peak of a preference is interpreted as a "Walrasian demand" under a fixed price. Another possible interpretation is a "task-sharing problem". Suppose that there is a group of agents that is involved in a production process. To complete the task, a fixed amount of (homogeneous) work is needed. Finally, each agent receives a piece of output according to his contribution. In this circumstance, it is natural to assume that each agent has a single-peaked preference over the space of quantity of work (the real line).[1]

In this paper, we call a mapping that associates each list of preferences with an allocation of resource a *resource allocation rule*, or simply a *rule*. If there is only one commodity, Sprumont (1991) presents a characterization of a resource allocation rule called *the uniform rule*.[2] Under the uniform rule, the same amount of the commodity is allotted to everyone except people whose peaks are small enough if excess demand exists or large enough if excess supply exists. He proves that the uniform rule is the only rule that satisfies three axioms: *strategy-proofness*, *Pareto efficiency*, and *anonymity*. *Strategy-proofness* means that announcing their true preferences is a dominant strategy for each agent in the game of stating their preferences. *Pareto efficiency* means that given a preference profile, the rule should select a Pareto efficient allocation for the reported preference profile. *Anonymity* says that the rule is independent of the "names" of the agents.

Sprumont's theorem essentially depends on the assumption that there is only one commodity. If the number of commodities is greater than one, we may naturally extend the uniform rule. The extended rule is referred to as *the generalized uniform rule*. The generalized uniform rule is defined by applying the single commodity uniform rule commodity by commodity.

It is easy to show that the generalized uniform rule is strategy-proof.[3] However, as shown in Example 1 below, the rule *violates Pareto efficiency*.

*Example 1* There are two agents and two commodities. Let $\Omega_1$ and $\Omega_2$ be the amounts of commodities 1 and 2. Figure 1 is an Edgeworth Box. In Fig. 1, $p(R^1)$ and $p(R^2)$ designate the peaks of Mr. 1 and Mr. 2's preferences, respectively. The middle point $(\frac{\Omega_1}{2}, \frac{\Omega_2}{2})$ is the allocation where equal amounts of commodities are allotted to each agent. Since for each commodity $j = 1, 2$, both agents have peaks greater than $\frac{\Omega_j}{2}$, the generalized uniform rule assigns equal amounts of commodities to both. This allocation is given by $(\frac{\Omega_1}{2}, \frac{\Omega_2}{2})$. However, if their indifference curves through $(\frac{\Omega_1}{2}, \frac{\Omega_2}{2})$ can be drawn as in Fig. 1, there is room for Pareto improvement. (For example, $\mathbf{x}$ is better than $(\frac{\Omega_1}{2}, \frac{\Omega_2}{2})$ for both agents.)

The literature on *strategy-proofness* in economic environments with classical preferences has uncovered the incompatibility among *strategy-proofness, Pareto efficiency* and some weak fairness notions. For example, in pure exchange economies with two

[1] These examples are taken from Sprumont (1991).

[2] The rule was originally introduced by Benassy (1982).

[3] It also satisfies *anonymity*.

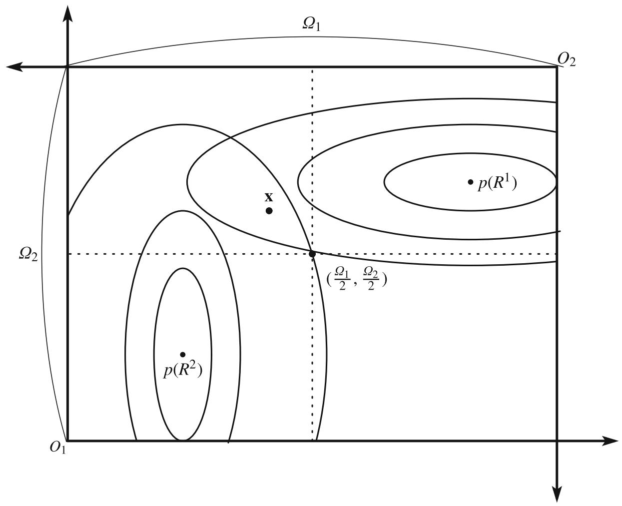


**Fig. 1** The generalized uniform rule violates *Pareto efficiency*

agents and two commodities, no rule is strategy-proof, Pareto efficient, and individually rational (Hurwicz 1972).[4] In two agents and $m$ commodities economies, no rule is strategy-proof, Pareto efficient, and non-dictatorial (Zhou 1991).[5] In $n$ agents and $m$ commodities economies, no rule is strategy-proof, Pareto efficient, and individually rational (Serizawa 2002). Finally, in $n$ agents and $m$ commodities economies, strategy-proof and Pareto efficient rules cannot guarantee a minimum consumption: for arbitrarily small $\varepsilon > 0$, there exists a preference profile such that under the rule and the preference profile, there is an agent whose consumption has a Euclidean norm smaller than $\varepsilon$ (Serizawa and Weymark 2003).[6]

The same kind of incompatibility exists in economies with multiple commodities and single-peaked preferences. Amorós (2002) points out that in economies with two agents and $m$ commodities, where each agents' preference is single-peaked, no rule is strategy-proof, Pareto efficient, and non-dictatorial.

Facing with the impossibility, the best one can do is to drop or relax an axiom. In fact, Amorós (2002) resolved this difficulty along this line. That is, he relaxed *Pareto efficiency* to the axiom of *same-sidedness*. This axiom requires that for each commodity, the amount of the commodity received by everyone be located on the same side of the agent's own peak.[7] His main theorem says that if the number of agents is *two*

[4] Individual rationality requires that the selected allocation be better than or indifferent to initial endowment for each agent. In this paper, we assume that some amount of commodities is initially owned by society, not by each individual.

[5] A rule is dictatorial if there exist an agent who always receives the most desirable allotment according to his preference. If a rule has no dictator, then we say that the rule is non-dictatorial.

[6] It is known that this kind of incompatibility does not hold when the domain of a rule is the set of Leontief preferences. See Nicolò (2004) and Li and Xue (2012).

[7] That is, if the quantity of a commodity received by an agent is greater than or equal to his own peak amount, then the quantities of the commodity received by other agents should be greater than or equal to

and the number of commodities is greater than or equal to two, *the generalized uniform rule is the unique rule satisfying no-envy, strategy-proofness, and same-sidedness.*[8] Since *same-sidedness* is a straightforward extension of Sprumont's efficiency concept, Amorós' theorems may be understood as a multi-commodity version of Sprumont's characterization.[9]

In the present paper, however, we study the incompatibility from a much different point of view. To state our approach, let us discuss how *strategy-proofness* is treated in the present paper. We regard *strategy-proofness* as an indispensable axiom for a rule throughout this paper. For designing a well-worked rule, one of the most problematic point caused by a strategic behavior of agents is that it may destroy the normative requirements like efficiency and fairness. To implement such normative requirements without being interfered by a strategic behavior, *strategy-proofness* should be imposed on a rule. The importance of *strategy-proofness* is emphasized by us as well as many authors including Amorós (2002), Morimoto et al. (2012), and Adachi (2010). Although both we and these authors recognize the importance of the *strategy-proofness*, our approach is unique in that we introduce new second-best efficiency concepts that are much different from *same-sidedness*.[10] More precisely, letting $f$ and $g$ be any rules, $f$ is said to Pareto-dominate $g$ if $f^i(\mathbf{R})$ is at least as good as $g^i(\mathbf{R})$ for each $i$ and each preference profile $\mathbf{R}$. We consider *the set* $\Gamma_{SP}$ *of all strategy-proof rules* and propose two concepts of second-best efficiency. The first one is *weak second-best efficiency among strategy-proof rules (WSESP)*. A strategy-proof rule $f_0$ is *WSESP* if for any strategy-proof rule $f_1$ that Pareto-dominates $f_0$, $f_0$ Pareto-dominates $f_1$. The second concept is called *strong second-best efficiency among strategy-proof rules (SSESP)*. A strategy-proof rule $f_0$ is *SSESP* if for any strategy-proof rule $f_1$ that Pareto-dominates $f_0$, $f_0 = f_1$. We will present a characterization of the generalized uniform rule with *WSESP* in a two agents and $m$ commodities setup and compare the result with the previous authors' characterizations in Sect. 3.

In the proof of the main results (Theorems 1, 2, 3, 4), the *option set* of each agent plays an important role.[11] For the $i$-th agent, given a preference profile $R^{-i} = (R^1, \ldots, R^{i-1}, R^{i+1}, \ldots, R^n)$ of the other agents, the option set of the $i$-th agent under a rule $f$ is the set of bundles that may be assigned to him by $f$ if the

Footnote 7 continued
their own peak amounts, and *vice versa*. If the number of commodity is one, *same-sidedness* is equivalent to *Pareto efficiency*. However, if the number of commodity is greater than one, *same-sidedness* is *not* a sufficient condition for *Pareto efficiency*. In this sense, it is strictly weaker than *Pareto efficiency*. Amorós (2002) called the axiom *Condition E* (CE).

[8] Theorem 2 in Amorós (2002). In his Theorem 3, he replaces no-envy with *weak anonymity*. Recently, Morimoto et al. (2012) and Adachi (2010) extended Amorós' result to the case of arbitrary number of agents.

[9] As noted earlier, *same-sidedness* is equivalent to *Pareto efficiency* if the number of commodity is one. In his proof, Sprumont (1991) needs only geometric property from *same-sidedness*. Hence, in multiple-commodity economies, only *same-sidedness* is required for extending Sprumont's characterization.

[10] Precisely speaking, Morimoto et al. (2012) do not adopt *same-sidedness* directly. But the combination of some axioms they adopt implies *same-sidedness*. See Lemma 1 in Morimoto et al. (2012).

[11] The notion of the option set is effectively used in many papers including Barberà (1983) and Barberà and Peleg (1990). See also Bordes et al. (2011).

other agents announce the preference profile $R^{-i}$. In the set $\Gamma_{\mathrm{SP}}$ of all strategy-proof rules, we prove that for any rules $f$ and $g \in \Gamma_{\mathrm{SP}}$, $f$ Pareto-dominates $g$ *if and only if* the option set of $g$ is included in that of $f$ for each agents (Lemma 4). Thus, on the domain $\Gamma_{\mathrm{SP}}$, the relation of Pareto-domination is equivalent to the relation of set theoretic inclusion between option sets. This observation is important in the proofs. For example, in Theorem 1, we prove the existence of a Pareto-undominated rule (a WSESP rule). In the proof of the theorem, it is enough to show that there exists a *maximal* option set with respect to the inclusion relation in the collection of option sets.

This paper consists of four sections. In Sect. 2, we present the model and describe our axioms. Moreover, we examine the existence of strategy-proof, efficient, and equitable allocation rules in an $n$ agents and $m$ commodities setup. In Sect. 3, a new characterization of the generalized uniform rule is given in a two agents and $m$ commodities setup. Section 4 concludes. All proofs are relegated to "Appendix".

## 2 Second-best efficiency of resource allocation rules

### 2.1 Single-peaked preferences with multiple commodities

Let $N = \{1, \ldots, n\}$ be the set of agents. Let $M = \{1, \ldots, m\}$ be the set of commodities. All commodities are perfectly divisible. The bundle $\Omega = (\Omega_1, \ldots, \Omega_m) \in \mathbb{R}^m_{++}$ denotes a social endowment of the commodities.[12] Let $B$ denote the set of feasible allocations. Since we do not allow free disposal, $B = \left\{\mathbf{x} = (x^1, \ldots, x^n) \in (\mathbb{R}^m_+)^n \mid \sum_{i=1}^n x^i = \Omega\right\}$. The preferences of each agent are given by a complete, transitive, continuous, and strictly convex binary relation on $\prod_{j=1}^m [0, \Omega_j]$.[13]

**Definition 1** A preference $R$ is a (multidimensional) single-peaked preference if there exists $p(R) \in \prod_{j=1}^m [0, \Omega_j]$ such that for each $x, x' \in \prod_{j=1}^m [0, \Omega_j]$ with $x \neq x'$,

$$\left[\forall j \in M, x'_j \leq x_j \leq p_j(R) \text{ or } p_j(R) \leq x_j \leq x'_j\right] \Rightarrow x\ P(R)\ x'.$$

Let $\mathcal{R}$ be the class of single-peaked preferences.[14] We call an element of $\mathcal{R}^N$ a *preference profile,* or simply a *profile.* For each profile $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$, and

[12] The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural and real numbers, respectively. Let $\mathbb{R}_+$ be the set of non-negative numbers and let $\mathbb{R}_{++}$ be the set of positive real numbers.

[13] For each preference $R$, $P(R)$ and $I(R)$ denote the asymmetric part of $R$ and the symmetric part of $R$, respectively.

[14] In Introduction, we give two interpretations of single-peaked preferences. These interpretations are due to Sprumont (1991), and they are meaningful even in a multi-dimensional setup as well as a one-dimensional setup. In addition to these interpretations, multi-dimensional single-peaked preferences have an inherent and important interpretation. It is about a design of a risk-sharing rule. To see this, let us consider an example of the task-sharing under uncertainty.

Suppose that a group of agents participates in a production process, say cultivating rice, and suppose that there are finitely many states of nature (for example, hot summer and cold summer). Since crops of rice field depend on weather, the quantity of harvested rice may vary depending on the states of nature. Each agent receives a portion of the product according to their own contribution and a realized state.

Since this is an extension of the task-sharing example in Introduction to a case with quantity uncertainty, it is natural to assume that his inter-state utility representation $u^i(x^i_1, x^i_2, \ldots, x^i_m)$ is $m$-dimensionally each $i \in N$, the subprofile obtained by removing $i$'s preference is denoted by $\mathbf{R}^{-i}$; that is, $\mathbf{R}^{-i} = (R^1, \ldots, R^{i-1}, R^{i+1}, \ldots, R^n)$. It is convenient to write the profile $(R^1, \ldots, R^{i-1}, \hat{R}^i, R^{i+1}, \ldots, R^n)$ as $(\hat{R}^i; \mathbf{R}^{-i})$. A *rule* is a mapping from $\mathcal{R}^N$ to $B$. Let $\Gamma$ denote the set of all rules.

## 2.2 Basic axioms

In this subsection, we introduce our axioms. Let $f$ be the generic notation for rules. The first axiom is a quite standard requirement of efficiency. It says that no one can be made better off at the selected allocation without someone else being made worse off.

*Pareto efficiency (PE)*: for each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$, there is no $\mathbf{x} = (x^1, \ldots, x^n) \in B$ such that i) $x^i \ R^i \ f^i(\mathbf{R})$ for each $i \in N$; and ii) $x^i \ P(R^i) \ f^i(\mathbf{R})$ for some $i \in N$.

Next, we introduce three fairness axioms. In our model, the social endowment is interpreted as the resource jointly owned by all agents in the society. Hence, it is natural to consider that the welfare level at the equal division should be guaranteed in selected allocations. The following axiom embodies this idea.

*Egalitarian rationality (ER)*[15]: for each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$ and each $i \in N$, $f^i(\mathbf{R}) \ R^i \ \frac{\Omega}{n}$.

The second fairness axiom requires that no agent prefer another agent's allotment to his own in selected allocations.

*No-envy (NE)*: For each $\mathbf{R} \in \mathcal{R}^N$, and each $i, j \in N$, $f^i(\mathbf{R}) \ R^i \ f^j(\mathbf{R})$.

The third fairness axiom says that if two agents have the same preference, then their welfare level should be the same.

*Symmetry (SY)*: For each $\mathbf{R} \in \mathcal{R}^N$, and each $i, j \in N$, if $R^i = R^j$, then $f^i(\mathbf{R}) \ I(R^i) \ f^j(\mathbf{R})$.

Note that *NE* implies *SY*. There is no logical relationship between *ER* and *NE* in general. For economies with $n = 2$, it is known that *ER* is a stronger requirement than *NE*. See, for example, Thomson (2011).

Now, we introduce an incentive compatibility axiom called *strategy-proofness*. Since Hurwicz (1972), it has been recognized by economists that the price mechanism

Footnote 14 continued
single-peaked, where $x^i_s$ is the quantity of rice allocated to the $i$-th agent when the $s$-th state of nature is revealed. See Ju (2005) for the case of monotonic preferences.

[15] This axiom is sometimes referred to as *the equal division lower bound*.

can be manipulated by a strategic behavior of an agent. More seriously, in some preference profiles, a strategic behavior of an agent can destroy the efficiency of the price mechanism. That is, the selected allocation at the false preference profile can be inefficient in terms of the true preference profile. As mentioned in Introduction, throughout this paper, we keep assuming *strategy-proofness* because we need to implement normative axioms such as fairness without being interfered by a strategic behavior of an agent. The axiom requires that no agent can ever be better off by misreporting his preference.

*Strategy-proofness (SP)*: for each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, $f^i(\mathbf{R})\ R^i\ f^i(\hat{R}^i; \mathbf{R}^{-i})$.

Let $\Gamma_{\text{SP}}$ denote the set of all strategy-proof rules.

## 2.3 Second-best efficiency

As we saw in Introduction, many authors have shown the incompatibility among *SP*, *PE* and some fairness notions in pure exchange economies (Hurwicz 1972; Zhou 1991; Serizawa 2002; Serizawa and Weymark 2003). This kind of difficulty arises in our single-peaked preference model with multiple commodities, too. Amorós (2002) shows that every strategy-proof and Pareto-efficient rule is dictatorial when $n = 2$ in the single-peaked preference models with multiple commodities.[16] Note that *ND* is a much weaker axiom than the three fairness notions appeared in the previous subsection. This result parallels the negative result shown by Zhou (1991) in two agents pure exchange economies. Although we do not pursue this line of investigation in this paper, Amorós' impossibility theorem suggests that the same negative result could be obtained even if the number of agents is greater than two.

Facing the incompatibility among *SP*, *PE* and some fairness notions, to design an plausible rule in our setup, we must give up at least one of them. As we have noticed in Introduction, we keep assuming *SP* because it is the condition that guarantees a rule to behave as intended. In this paper, we relax *PE* rather than fairness notion. Otherwise we cannot help accepting dictatorship as the Amorós' theorem suggested. In the rest of this subsection, we address the two following natural questions:

**Question 1**. How can we relax the concept of *PE* while preserving *SP*? That is, what kind of second-best efficiency concepts do we have?

**Question 2**. Is there a rule satisfying *SP* and *ND* in addition to second-best efficiency?

As an answer to Question 1, we introduce two second-best efficiency concepts. The axioms are based on Pareto-domination in the set of rules. To formalize them,

[16] An agent $i \in N$ is a dictator of a rule $f$ *if* for each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$, $f^i(\mathbf{R}) = p(R^i)$. Obviously, a rule has at most one dictator. A rule $f$ is dictatorial *if* $f$ has a dictator. The following is the weakest fairness condition in this paper.
*Non-dictatorship (ND)* : $f$ does not have a dictator.

we define the Pareto-dominance relation. "A rule $f$ Pareto-dominates a rule $g$" means that for each preference profile and each agent, the consumption bundle selected by $f$ is preferred or is indifferent to the one selected by $g$.

**Definition 2**[17] For each $f, g \in \Gamma$,

$$f \text{ dom } g \;\Leftrightarrow \forall \mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N, \forall i \in N,\; f^i(\mathbf{R})\; R^i\; g^i(\mathbf{R}).$$

Using the dom relation, we obtain the following equivalent expression of *PE*. This expression clarifies that *PE* requires a rule to be a maximal element of $\Gamma$ preordered by dom.

*Pareto efficiency (PE)* : for each $g \in \Gamma$, $g$ dom $f \Rightarrow f$ dom $g$.

Now we move on to our second-best efficiency concepts. The first one (*i*) requires that a rule be a maximal element of $\Gamma_{\text{SP}}$ preordered by dom. This is a natural requirement of efficiency as long as we keep assuming *SP* because if we have two *SP* rules $f$ and $g$ satisfying $f$ dom $g$ and not $g$ dom $f$, then the agents in the society unanimously choose $f$. The second one (*ii*) is more demanding. It requires that a rule be a maximal element of $\Gamma_{\text{SP}}$ preordered by dom and that no other rule be welfare-equivalent to it.

*(i) Weak second-best efficiency among all strategy-proof rules (WSESP)*: $f \in \Gamma_{\text{SP}}$ and for each $g \in \Gamma_{\text{SP}}$, $g$ dom $f \Rightarrow f$ dom $g$.

*(ii) Strong second-best efficiency among all strategy-proof rules (SSESP)*: $f \in \Gamma_{\text{SP}}$ and for each $g \in \Gamma_{\text{SP}}$, $g$ dom $f \Rightarrow f = g$.[18]

The first one is logically implied by *PE* if a rule also satisfies *SP* because the axiom is a natural restriction to $\Gamma_{\text{SP}}$. Clearly, *SSESP* also implies *WSESP*. As shown in the next section, the generalized uniform rule not only satisfies *WSESP*, but also satisfies *SSESP*. This fact is crucial to our characterization.

Next, we answer Question 2. The following theorem states that for any rule $f$ which satisfies *SP*, we have a *WSESP* rule that dominates $f$. This theorem will be used to prove that our characterization is tight in Sect. 3.3. Note that in the following theorems, the number of agents is arbitrary.

**Theorem 1** *For any $f \in \Gamma_{SP}$, there exists a rule $f_0 \in \Gamma_{SP}$ that is weakly second-best efficient among all strategy-proof rules (WSESP) such that $f_0$ dom $f$.*

As a consequence of relaxing *PE*, we obtain the following positive result. Since *ND* is implied by *ER*, the answer to Question 2 is yes as far as *WSESP* is concerned.[19]

[17] Note that dom is reflexive and transitive. Hence, it is a preorder on $\Gamma$. But it is not an order on $\Gamma$ in general. That is, $f$ dom $g$ and $g$ dom $f$ do not necessarily imply $f = g$. If $f$ dom $g$ and $g$ dom $f$, then we say $f$ and $g$ are equivalent with respect to the welfare. Note also that dom is not complete.

[18] Sasaki (2003) first introduced *SSESP*. He called this condition $\Lambda_{\text{SP}}$-efficiency.

[19] Remember that no rule satisfies *SP, PE* and *ND* when $n = 2$.

**Theorem 2** *There exists a strategy-proof rule satisfying egalitarian rationality (ER) and weak second-best efficiency among all strategy-proof rules (WSESP).*

## 3 A new characterization of the generalized uniform rule

### 3.1 The generalized uniform rule

One of the purposes of this paper is to characterize the following rule, which is known as *the generalized uniform rule*.

**Definition 3** For each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$, each $j \in M$, each $i \in N$,

$$U^i_j(\mathbf{R}) = \begin{cases} \min\{p_j(R^i), \lambda_j(\mathbf{R})\} & \text{if } \sum_{i=1}^n p_j(R^i) \geq \Omega_j, \\ \max\left\{p_j(R^i), \mu_j(\mathbf{R})\right\} & \text{if } \sum_{i=1}^n p_j(R^i) \leq \Omega_j, \end{cases}$$

where $\lambda_j(\mathbf{R})$ solves the equation $\Omega_j = \sum_{i=1}^n \min\{p_j(R^i), \lambda_j(\mathbf{R})\}$ and $\mu_j(\mathbf{R})$ solves the equation $\Omega_j = \sum_{i=1}^n \max\{p_j(R^i), \mu_j(\mathbf{R})\}$.

### 3.2 A characterization of the generalized uniform rule

The following theorem describes an efficiency aspect of the generalized uniform rule.

**Theorem 3** *Let $n = 2$. If a rule satisfies strategy-proofness (SP) and same-sidedness, then it satisfies strong second-best efficiency among all strategy-proof rules (SSESP). In particular, the generalized uniform rule satisfies strong second-best efficiency among all strategy-proof rules (SSESP).*[20]

In the previous subsection, we point out that for any $n \geq 2$, there exists a rule satisfying *WSESP* and *ER*. Since the generalized uniform rule satisfies *ER*, by Theorem 3, it is one of the rules satisfying *WSESP* and *ER* when $n = 2$.

Now, we ask whether there exists a rule satisfying *WSESP* and *ER* other than the generalized uniform rule. The answer is yes. An example of such a rule is provided in Example 4. However, Corollary 1 shows that if we impose the following condition in addition to *WSESP* and *ER*, then the generalized uniform rule is the unique rule that satisfies these three axioms. Moreover, we can relax *ER* for the uniqueness result.

*Weak peak-onliness (WP)*[21]: for each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$, each $i \in N$, each $\tilde{R}^i \in \mathcal{R}$,

[20] In the division problem of indivisible goods and money, Ohseto (2006) specifies the strategy-proof and no-envy rules that are not dominated by any other strategy-proof and no-envy rules. In the division problem of indivisible goods, Kesten and Yazıcı (2012) finds the strategy-proof and no-envy rule that dominates any other strategy-proof and no-envy rules. Theorem 3 says that, in our setting, the generalized uniform rule is not dominated by any other (no-envy or not) strategy-proof rules when $n = 2$. Klaus (2008) proves that the coordinate-wise core rule satisfies SSESP in the multiple-type housing markets.

[21] This axiom is sometimes referred to as *own peak only*.

$$p(R^i) = p(\tilde{R}^i) \Rightarrow f^i(\mathbf{R}) = f^i(\tilde{R}^i; \mathbf{R}^{-i}).$$

**Theorem 4** *Let $n = 2$. The generalized uniform rule is the only rule that satisfies weak second-best efficiency among all strategy-proof rules (WSESP), symmetry (SY), and weak peak-onliness (WP).*

**Corollary 1** *Let $n = 2$. The generalized uniform rule is the only rule that satisfies weak second-best efficiency among all strategy-proof rules (WSESP), egalitarian rationality (ER), and weak peak-onliness (WP).*

Now, let us compare Theorem 4 with the Amorós' characterizations of the generalized uniform rule. By Theorem 3 and Lemma 3 in Amorós (2002), *SP* and *same-sidedness* imply *WSESP* and *WP*. Since his fairness notions (*weak anonymity* and *no-envy*) imply *symmetry*, the Amorós' characterizations are also corollaries of Theorem 4.[22]

### 3.3 Independence of axioms in Theorem 4

In this subsection, we show that Theorem 4 is tight. That is, dropping any one of the three axioms leads to other rules.

*Example 2* An example of a rule that satisfies both ER and WP, but not WSESP is the equal division rule, E, defined as follows: for each $\mathbf{R} \in \mathcal{R}^N$, $E(\mathbf{R}) = (\frac{\Omega}{2}, \frac{\Omega}{2})$. Obviously $E$ satisfies ER and WP but $E$ not WSESP because $U$ dom $E$ but $E$ does not dominate $U$.

*Example 3* Examples of rules that satisfy both WP and WSESP, but not ER are the priority rules. Let $D^{(i)}$ be the priority rule in which agent $i$ has priority defined as follows: for each $\mathbf{R} = (R^1, R^2) \in \mathcal{R}^N$, $D^{(i)i}(\mathbf{R}) = p(R^i)$. Since $D^{(i)}$ satisfies SP and PE, it satisfies WSESP. It is also clear that $D^{(i)}$ satisfies WP. However, clearly, $D^{(i)}$ does not satisfy ER.

*Example 4* A rule that satisfies both ER and WSESP, but not WP is $f_0$ below. Let $f$ be the rule defined as follows. For each $\mathbf{R} = (R^1, R^2) \in \mathcal{R}^N$,

$$f(\mathbf{R}) = \begin{cases} (\Omega, 0) & \text{if } \Omega \ R^1 \frac{\Omega}{2} \text{ and } 0 \ R^2 \frac{\Omega}{2}, \\ (\frac{\Omega}{2}, \frac{\Omega}{2}) & \text{otherwise.} \end{cases}$$

Obviously $f$ satisfies SP and ER.

First, we show that $U$ does not dominate $f$. By Lemma 6 of Appendix, there exist $\tilde{R}^1, \tilde{R}^2 \in \mathcal{R}$ such that $p(\tilde{R}^1) = p(\tilde{R}^2) = (\Omega_1, \ldots, \Omega_{m-1}, 0)$ and $\Omega \ P(\tilde{R}^1)\frac{\Omega}{2}$ and $0 \ P(\tilde{R}^2)\frac{\Omega}{2}$. Then, $f(\tilde{R}^1, \tilde{R}^2) = (\Omega, 0)$. Since $U(\tilde{R}^1, \tilde{R}^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$, $U$ does not dominate $f$.

[22] Morimoto et al. (2012) and Adachi (2010) offer characterizations of the generalized uniform rule for economies with arbitrary number of agents. If we limit ourselves for the case with $n = 2$, their results are also implied by Theorem 4.

By Theorem 1, there is $f_0 \in \Gamma_{SP}$ such that $f_0$ is WSESP and $f_0$ dom $f$. Since $f$ satisfies ER and dom is transitive, $f_0$ satisfies ER. Since $f_0$ dom $f$, $f_0^1(\tilde{R}^1, \tilde{R}^2)\tilde{R}^1 \Omega\; P(\tilde{R}^1)\frac{\Omega}{2}$ and $f_0^2(\tilde{R}^1, \tilde{R}^2)\tilde{R}^2\; 0\; P(\tilde{R}^2)\frac{\Omega}{2}$. Since $U(\tilde{R}^1, \tilde{R}^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$, $f_0 \neq U$. This means that $f_0$ does not satisfy WP because if it does, then by Theorem 4, it is the uniform rule ($f_0 = U$), a contradiction.

By Examples 2, 3 and 4, we have shown that the three axioms in Theorem 4 are independent.

## 4 Concluding remarks

In many models with multiple commodities including pure exchange economies, it is known that no allocation rule satisfies *SP*, *PE*, and some fairness notions. This negative result is valid for our setup with single-peaked preferences and multiple commodities, especially in the two-agent case. In contrast to this fact, in this paper, we proposed second-best efficiency concepts and showed that *WSESP* is compatible with *SP* and *ER* in $n$ agents $m$ commodities economies (Theorem 2).

In addition, we showed that in two agents $m$ commodities economies, the generalized uniform rule is the only rule that satisfies *ER, WP* and *WSESP* (Corollary 1). As a conclusion, we present some open questions concerning second-best efficiency.

Whether the generalized uniform rule in economies with $n$ agents and $m$ commodities satisfies *WSESP* is still open.

In pure exchange economies with non-satiated preferences, we may raise a similar question. Barberà and Jackson (1995) investigate the exchange economies in which the prices are rigid. To study whether their strategy-proof allocation rules are second-best efficient seems to be a very important issue. In fact, Sasaki (2006) shows that fixed-price trading satisfies *SSESP* in a two agents, two commodities pure exchange economy.[23] Extending Sasaki's result to the $n$ agents $m$ commodities setting would be interesting. In addition, Barberà and Jackson (1995) investigate a class of rules called fixed-proportion trading and characterize the rules with *SP* and *individual rationality* in a model with 2 agents and $m$ commodities. However, the class of fixed-proportion trading rules includes some inefficient rules. (For example, the trivial rule, which always assigns the initial endowment, belongs to the class.) With the second-best efficiency concepts proposed in this paper, we may characterize a subclass of fixed-proportion trading rules.

More generally, as discussed in Introduction, it is known that there are several impossibility theorems suggesting the existence of a trade-off between *SP* and *PE* in various kinds of resource allocation problems. If the requirement of *PE* is weakened as done in the present paper, the same kind of positive results may be obtained in different models.

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[23] See Barberà and Jackson (1995).

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## Appendix: Proofs

### A.1 Proofs of Theorems 1 and 2

To prove Theorems 1 and 2, we need some lemmas concerning metric spaces. Let $(\mathbf{X}, d)$ be a metric space. Let $\bar{d} : \mathbf{X} \times (2^{\mathbf{X}} \backslash \{\emptyset\}) \to \mathbb{R}$ be the function defined by $\bar{d}(x, A) = \inf\{d(x, a) \mid a \in A\}$ for each $x \in \mathbf{X}$ and each $A \in 2^{\mathbf{X}} \backslash \{\emptyset\}$.[24] Let $\mathcal{K}(\mathbf{X})$ be the set of all non-empty compact subsets of $(\mathbf{X}, d)$.[25] Finally, let $d_H : \mathcal{K}(\mathbf{X}) \times \mathcal{K}(\mathbf{X}) \to \mathbb{R}$ be the function defined by

$$d_H(A, B) = \max\{\max_{a \in A} \bar{d}(a, B), \max_{b \in B} \bar{d}(b, A)\}.$$

It is well-known that $(\mathcal{K}(\mathbf{X}), d_H)$ is also a metric space. The metric $d_H$ is referred to as Hausdorff metric.

*Remark 1* Let $A, B \in \mathcal{K}(\mathbf{X})$. If $A \subseteq B, \bar{d}(a, B) = 0$ for each $a \in A$. Hence, $d_H(A, B) = \max_{b \in B} \bar{d}(b, A)$ for each $A, B \in \mathcal{K}(\mathbf{X})$ with $A \subseteq B$.

**Lemma 1** *Let $\mathcal{S} \subseteq \mathcal{K}(\mathbf{X})$ be totally ordered by $\subseteq$.*[26] *Let $C = cl_d(\cup_{S \in \mathcal{S}} S)$.*[27] *Then,*

$$\forall S_1, S_2 \in \mathcal{S}, [\{\exists x_0 \in C \text{ s.t. } \bar{d}(x_0, S_1) > \bar{d}(x_0, S_2)\} \Rightarrow S_1 \subseteq S_2].$$

*Proof* Suppose to the contrary, that $S_1 \not\subseteq S_2$. Since $\mathcal{S}$ is totally ordered by $\subseteq$, $S_2 \subsetneq S_1$. Then,

$$\begin{aligned} \bar{d}(x_0, S_1) &= \min\{d(x_0, y) \mid y \in S_1\} \\ &\leq \min\{d(x_0, y) \mid y \in S_2\} \quad (\because S_2 \subsetneq S_1) \\ &= \bar{d}(x_0, S_2). \end{aligned}$$

This is a contradiction. □

**Lemma 2** *Let $(\mathbf{X}, d)$ be a compact metric space. Let $\mathcal{S} \subseteq \mathcal{K}(\mathbf{X})$ be non-empty and totally ordered by $\subseteq$. Let $C = cl_d(\cup_{S \in \mathcal{S}} S) \in \mathcal{K}(\mathbf{X})$. Then, there is a sequence of compact sets in $\mathcal{S}$ that converges to $C$ with respect to $d_H$.*

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[24] Note that $\bar{d}(\cdot, A)$ is a continuous function from $\mathbf{X}$ to $\mathbb{R}$ when we fix $A \in 2^{\mathbf{X}} \backslash \{\emptyset\}$ arbitrarily.

[25] Note that $\bar{d}(x, A) = \min\{d(x, a) \mid a \in A\}$ when we restrict the domain of $\bar{d}$ within $\mathbf{X} \times \mathcal{K}(\mathbf{X})$.

[26] In general, $(\mathbf{Z}, \geq)$ is an *ordered set* if $\geq$ is a reflexive, transitive and anti-symmetric binary relation on $\mathbf{Z}$, where $\geq$ is reflexive if for each $x \in \mathbf{Z}$, $x \geq x$, and $\geq$ is anti-symmetric if for each $x, y \in \mathbf{Z}$, if $x \geq y$ and $y \geq x$, then $x = y$. An ordered set $(\mathbf{Z}, \geq)$ is *total* if for each $x, y \in \mathbf{Z}$, $x \geq y$ or $y \geq x$.

[27] Note that $\text{cl}_d(\cup_{S \in \mathcal{S}} S)$ denotes the closure of $\cup_{S \in \mathcal{S}} S$ in $(\mathbf{X}, d)$.

*Proof* If $C \in \mathcal{S}$, then the conclusion is trivial. We consider the case $C \notin \mathcal{S}$.

***Step 1*** For each $S' \in \mathcal{S}$, there exists $S'' \in \mathcal{S}$ such that $d_H(S'', C) < \frac{d_H(S', C)}{2}$.

Let $S' \in \mathcal{S}$. Since $d_H$ is a metric and $S' \neq C$, $d_H(S', C) > 0$. Let $\eta := d_H(S', C)$. Let $H = \{x \in C \mid \bar{d}(x, S') \geq \frac{\eta}{2}\}$. Note that $H$ is non-empty because $\bar{d}(x^*, S') = \eta$ for some $x^* \in C$. Since $H$ is the inverse image of $[\frac{\eta}{2}, +\infty)$ under the continuous function $\bar{d}(\cdot, S')$, it is closed. Hence, because $H \subseteq C \in \mathcal{K}(\mathbf{X})$, $H$ is compact.

By compactness of $H$, for an open cover $\{B(x, \frac{\eta}{4})\} \mid x \in H\}$ of $H$, we may find a finite points set of $x_1, \ldots, x_h \in H$ such that $H \subseteq \cup_{i=1}^{h} B(x_i, \frac{\eta}{4})$.[28] Since $x_1, \ldots, x_h \in H \subseteq \mathrm{cl}_d(\cup_{S \in \mathcal{S}} S)$, then

$$\forall i \in \{1, \ldots, h\}, \exists S_i \in \mathcal{S} \text{ s.t. } S_i \cap B\left(x_i, \frac{\eta}{4}\right) \neq \emptyset.$$

Since $\{S_1, \ldots, S_h\}$ are totally ordered by $\subseteq$, there is the greatest element in $\{S_1, \ldots, S_h\}$. We call it $S''$.

In the following, we show $d_H(S'', C) < \frac{\eta}{2}$. To this end, because of Remark 1, it is sufficient to show that for each $x \in C$, $\bar{d}(x, S'') < \frac{\eta}{2}$.

Let $x \in C$. First, suppose that $x \in H$. Then,

$$\exists i_x \in \{1, \ldots, h\} \text{ s.t. } x \in B\left(x_{i_x}, \frac{\eta}{4}\right).$$

Since $S'' \cap B(x_{i_x}, \frac{\eta}{4}) \neq \emptyset$, there is $y \in S'' \cap B(x_{i_x}, \frac{\eta}{4})$. By the triangle inequality,

$$\begin{aligned} d(x, y) &\leq d(x, x_{i_x}) + d(x_{i_x}, y) \\ &< \tfrac{\eta}{4} + \tfrac{\eta}{4} \\ &= \tfrac{\eta}{2}. \end{aligned}$$

By the definition of $\bar{d}$, we have $\bar{d}(x, S'') < \frac{\eta}{2}$.

Note that we have shown that $\bar{d}(x^*, S') = \eta > \frac{\eta}{2} > \bar{d}(x^*, S'')$. Hence, by Lemma 1, $S' \subseteq S''$.

Next suppose that $x \notin H$. Then,

$$\begin{aligned} \tfrac{\eta}{2} &> \bar{d}(x, S') \ (\because x \notin H) \\ &= \min\{d(x, y) \mid y \in S'\} \\ &\geq \min\{d(x, y) \mid y \in S''\} \ (\because S' \subseteq S'') \\ &= \bar{d}(x, S''). \end{aligned}$$

This completes the proof of Step 1.

***Step 2*** *Applying the Axiom of Choice.*

[28] Note that $B(x, \varepsilon)$ denotes the open ball centered at $x$ with a radius $\varepsilon$.

By Step 1 and the axiom of choice, there is a function $\Phi : \mathcal{S} \to \mathcal{S}$ such that

$$\forall S \in \mathcal{S}, d_H(\Phi(S), C) < \frac{d_H(S, C)}{2}.$$

Let $S \in \mathcal{S}$. Let $S_1 \equiv S$. For each $k \geq 2$, let $S_k \equiv \Phi(S_{k-1})$. Since for each $k \in \mathbb{N}$, $d_H(S_k, C) < \frac{d_H(S,C)}{2^{k-1}}$, $\{S_k\}_{k\in\mathbb{N}}$ satisfies $d_H(S_k, C) \to 0$ as $k \to +\infty$. □

*Remark 2* The sequence obtained in Lemma 2 satisfies

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots.$$

**Definition 4** Let $f \in \Gamma$ and $i \in N$. For each $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$,

$$B^{f^i_{\mathbf{R}^{-i}}} \equiv \left\{ x \in \prod_{j=1}^{m} [0, \Omega_j] \mid \exists R^i \in \mathcal{R} \text{ s.t. } f^i(R^i; \mathbf{R}^{-i}) = x \right\}.$$

We call $B^{f^i_{\mathbf{R}^{-i}}}$ the option set of agent $i$ under $f$ and $\mathbf{R}^{-i}$.

Let $\tau(R, Y) = \{x \in Y \mid \forall y \in Y, xRy\}$ for each $Y \subseteq \prod_{j=1}^{m}[0, \Omega_j]$, each $R \in \mathcal{R}$. That is, $\tau(R, Y)$ denotes the best consumptions on $Y \subseteq \prod_{j=1}^{m}[0, \Omega_j]$ with respect to $R \in \mathcal{R}$.[29]

**Lemma 3** *Let* $f \in \Gamma$.

$$f \in \Gamma_{SP} \Leftrightarrow \forall \mathbf{R} \in \mathcal{R}^N, \forall i \in N,\ f^i(\mathbf{R}) \in \tau(R^i, B^{f^i_{\mathbf{R}^{-i}}}).$$

*Proof* Obvious. □

**Lemma 4** *Let* $f, g \in \Gamma_{SP}$.

$$f \ dom \ g \Leftrightarrow \forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, B^{g^i_{\mathbf{R}^{-i}}} \subseteq B^{f^i_{\mathbf{R}^{-i}}}.$$

*Proof* ($\Rightarrow$) Let $i \in N, \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$. Let $x \in B^{g^i_{\mathbf{R}^{-i}}}$. Let $R^i \in \mathcal{R}$ be such that $\tau(R^i, \prod_{j=1}^{m}[0, \Omega_j]) = x$. By Lemma 3, $g^i(R^i; \mathbf{R}^{-i}) = x$. Since $f$ dom $g$, $f^i(R^i; \mathbf{R}^{-i})\ R^i\ g^i(R^i; \mathbf{R}^{-i})$. Hence, $f^i(R^i; \mathbf{R}^{-i}) = x$. This implies $x \in B^{f^i_{\mathbf{R}^{-i}}}$.

($\Leftarrow$) Let $i \in N, \mathbf{R} \in \mathcal{R}^N$. By Lemma 3, $f^i(\mathbf{R}) \in \tau(R^i, B^{f^i_{\mathbf{R}^{-i}}})$ and $g^i(\mathbf{R}) \in \tau(R^i, B^{g^i_{\mathbf{R}^{-i}}})$. Since $B^{g^i_{\mathbf{R}^{-i}}} \subseteq B^{f^i_{\mathbf{R}^{-i}}}$, $f^i(\mathbf{R})\ R^i\ g^i(\mathbf{R})$. □

[29] If $f \in \Gamma_{\text{SP}}$, then $B^{f^i_{\mathbf{R}^{-i}}}$ is closed set in $\prod_{j=1}^{m}[0, \Omega_j]$ for each $i \in N, \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$. To see this, fix $i \in N$ and $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, then we have a one agent social choice function from $\mathcal{R}$ to $\prod_{j=1}^{m}[0, \Omega_j]$ induced by $f$. Note that $B^{f^i_{\mathbf{R}^{-i}}}$ is the range of the one agent social choice function. Applying Proposition 5 in Le Breton and Weymark (1999), we have the conclusion. See also Barberà and Peleg (1990).

For each $f, g \in \Gamma_{\mathrm{SP}}$, $f \sim g$ if and only if $f$ dom $g$ and $g$ dom $f$. Note that $\sim$ is an equivalence relation on $\Gamma_{\mathrm{SP}}$. For each $f \in \Gamma_{\mathrm{SP}}$, $[f]$ denotes the equivalence class of $f$ in the quotient set $\Gamma_{\mathrm{SP}}/\sim$.

**Definition 5**[30] For each $[f], [g] \in \Gamma_{\mathrm{SP}}/\sim$,

$$[f] \text{ Dom } [g] \Leftrightarrow f \text{ dom } g.$$

**Definition 6** Let $(\mathbf{Z}, \geq)$ be an ordered set. We say that $(\mathbf{Z}, \geq)$ is inductive if every non-empty totally ordered subset of $\mathbf{Z}$ has an upper bound.

The following lemma shows an important property of an inductive ordered set.

**Zorn's Lemma** *Let* $(\mathbf{Z}, \geq)$ *be an ordered set which is inductive and let* $x_0$ *be an element of* $\mathbf{Z}$. *Then*

$$\exists a \in \mathbf{Z} \text{ s.t. } a \geq x_0 \text{ and } \forall x \in \mathbf{Z}, \neg(x > a).$$

**Lemma 5** *The ordered set* $\left(\Gamma_{SP}/\sim, Dom\right)$ *is inductive.*

*Proof* Let $\mathcal{S} \subseteq \Gamma_{\mathrm{SP}}/\sim$. Suppose that $\left(\mathcal{S}, \text{Dom } |_{\mathcal{S}\times\mathcal{S}}\right)$ is totally ordered. We construct a rule $F : \mathcal{R}^N \to B$ and show that $[F]$ is an upper bound of $\mathcal{S}$. For each $i \in N$, $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, $C^{\mathbf{R}^{-i}} := \mathrm{cl}_d\left(\cup_{[f]\in\mathcal{S}} B^{f^i_{\mathbf{R}^{-i}}}\right)$.

***Step 1*** *Existence of* $\{[f_1], [f_2], \ldots, [f_k], \ldots\}$.

In this step, we show that

$$\exists\{[f_k]\}_{k\in\mathbb{N}} \text{ in } \mathcal{S} \text{ s.t. } \forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, B^{f^i_{k\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}. \tag{1}$$

Note that $\{B^{f^i_{k\mathbf{R}^{-i}}}\}_{k\in\mathbb{N}}$ is a sequence in $(\mathcal{K}(\prod_{j=1}^m [0, \Omega_j]), d_H)$. Hence, $B^{f^i_{k\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}$ means the convergence of the sequence with respect to the $d_H$ metric.

By Lemma 2, for each $i \in N$, and each $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$,

$$\exists\{[f_{ik}]\}_{k\in\mathbb{N}} \text{ in } \mathcal{S} \text{ s.t. } B^{f^i_{ik\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}.$$

By Remark 2, for each $\mathbf{R} = (R^1, \ldots, R^n) \in \mathcal{R}^N$,

$$\begin{array}{c}
B^{f^1_{11\mathbf{R}^{-1}}} \subseteq B^{f^1_{12\mathbf{R}^{-1}}} \subseteq B^{f^1_{13\mathbf{R}^{-1}}} \subseteq \cdots, \\
B^{f^2_{21\mathbf{R}^{-2}}} \subseteq B^{f^2_{22\mathbf{R}^{-2}}} \subseteq B^{f^2_{23\mathbf{R}^{-2}}} \subseteq \cdots, \\
\vdots \\
B^{f^n_{n1\mathbf{R}^{-n}}} \subseteq B^{f^n_{n2\mathbf{R}^{-n}}} \subseteq B^{f^n_{n3\mathbf{R}^{-n}}} \subseteq \cdots.
\end{array}$$

[30] Note that Dom is an order on $\Gamma_{\mathrm{SP}}/\sim$.

For each $k \in \mathbb{N}$, since $\{[f_{1k}], [f_{2k}], \ldots, [f_{nk}]\}$ is totally ordered by Dom, there is a greatest element among $[f_{1k}], [f_{2k}], \ldots, [f_{nk}]$ with respect to Dom. Let $[f_k]$ be the greatest element. We show that $\{[f_k]\}_{k\in\mathbb{N}}$ satisfies (1).

Let $i \in N$, and $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, $k \in \mathbb{N}$. Because $f_k$ dom $f_{ik}$, $B^{f^i_{ik\mathbf{R}^{-i}}} \subseteq B^{f^i_{k\mathbf{R}^{-i}}}$, we obtain $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{k\mathbf{R}^{-i}}}) \leq d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{ik\mathbf{R}^{-i}}})$. Since $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{ik\mathbf{R}^{-i}}}) \to 0$ as $k \to +\infty$, then $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{k\mathbf{R}^{-i}}}) \to 0$ as $k \to +\infty$.

***Step 2*** *Defining a function F*

Let $\{[f_k]\}_{k\in\mathbb{N}}$ be a sequence obtained in Step 1. For each $\mathbf{R} \in \mathcal{R}^N$, we obtain a sequence $\{f_k(\mathbf{R})\}_{k\in\mathbb{N}}$ in $B$. Since $B$ is compact, there exists a convergent subsequence $\{f_{\mathbf{R}(k)}(\mathbf{R})\}_{k\in\mathbb{N}}$.[31] Now, we define $F : \mathcal{R}^N \to B$ as follows : for each $\mathbf{R} \in \mathcal{R}^N$, $F(\mathbf{R}) = \lim_{k\to+\infty} f_{\mathbf{R}(k)}(\mathbf{R})$. Obviously, $F \in \Gamma$.

***Step 3*** *For each $i \in N$, $\forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, $B^{F^i_{\mathbf{R}^{-i}}} = C^{\mathbf{R}^{-i}}$.*

(i) First, we show $B^{F^i_{\mathbf{R}^{-i}}} \subseteq C^{\mathbf{R}^{-i}}$. Let $x^i \in B^{F^i_{\mathbf{R}^{-i}}}$. Then,

$$\exists \hat{R}^i \in \mathcal{R} \text{ s.t. } F^i(\hat{R}^i; \mathbf{R}^{-i}) = x^i.$$

Let $\hat{\mathbf{R}} = (\hat{R}^i; \mathbf{R}^{-i})$. By the definition of $F$, $F(\hat{\mathbf{R}}) = \lim_{k\to+\infty} f_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}})$. For each $k \in \mathbb{N}$, by *SP* of $f_{\hat{\mathbf{R}}(k)}$, $f^i_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}}) \in \tau(\hat{R}^i, B^{f^i_{\hat{\mathbf{R}}(k)\mathbf{R}^{-i}}}) \subseteq B^{f^i_{\hat{\mathbf{R}}(k)\mathbf{R}^{-i}}} \subseteq C^{\mathbf{R}^{-i}}$. Hence, $\{f^i_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}})\}_{k\in\mathbb{N}}$ is a convergent sequence in $C^{\mathbf{R}^{-i}}$. Since $C^{\mathbf{R}^{-i}}$ is closed in $\prod_{j=1}^m [0, \Omega_j]$, $x^i \in C^{\mathbf{R}^{-i}}$.

(ii) Next, we show $C^{\mathbf{R}^{-i}} \subseteq B^{F^i_{\mathbf{R}^{-i}}}$. Let $x^i \in C^{\mathbf{R}^{-i}}$. Let $R^i_d \in \mathcal{R}$ be the preference represented by the utility function $u_d$ defined by $u_d(z) = -\|x^i - z\|$. Let $\mathbf{R}_d = (R^i_d; \mathbf{R}^{-i})$. We show that $F^i(\mathbf{R}_d) = x^i$ by contradiction. Suppose that $x^i \neq F^i(\mathbf{R}_d) = \lim_{k\to+\infty} f^i_{\mathbf{R}_d(k)}(\mathbf{R}_d)$. Then, without loss of generality, we may assume

$$\exists \varepsilon > 0, \exists K_1 \in \mathbb{N} \text{ s.t. } \forall \ell \geq K_1, f^i_{\mathbf{R}_d(\ell)}(\mathbf{R}_d) \notin B(x^i, \varepsilon). \tag{2}$$

Since $B^{f^i_{k\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}$, there exists $K_2 \in \mathbb{N}$ such that $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{K_2\mathbf{R}^{-i}}}) < \varepsilon$. By Remark 1, $B^{f^i_{K_2\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon) \neq \emptyset$. Furthermore, by Remark 2,

$$\forall \ell \geq K_2, B^{f^i_{\ell\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon) \neq \emptyset. \tag{3}$$

Let $L = \max\{K_1, K_2\}$. By (2) and (3),

$$f^i_{\mathbf{R}_d(L)}(\mathbf{R}_d) \notin B(x^i, \varepsilon) \text{ and } B^{f^i_{\mathbf{R}_d(L)\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon) \neq \emptyset.$$

[31] $\mathbf{R}(\cdot)$ is an operator from $\mathbb{N}$ to $\mathbb{N}$ creating a subsequence. In general, there may exist more than one convergent subsequences. In this case, let us choose an arbitrary convergent subsequence and define it as $\{f_{\mathbf{R}(k)}(\mathbf{R})\}_{k\in\mathbb{N}}$.

Now let $y \in B^{f^i_{\mathbf{R}_{d(L)}\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon)$ and let $\tilde{R}^i_d \in \mathcal{R}$ be such that $\tau(\tilde{R}^i_d, \prod_{j=1}^m [0, \Omega_j]) = \{y\}$. Then, by Lemma 3, $f^i_{\mathbf{R}_{d(L)}}(\tilde{R}^i_d; \mathbf{R}^{-i}) = y$. This implies that $f_{\mathbf{R}_{d(L)}}$ is manipulable by $i$ at $(R^i_d; \mathbf{R}^{-i})$ via $\tilde{R}^i_d$, a contradiction. Hence $F^i(\mathbf{R}_d) = x^i$. This implies $x^i \in B^{F^i_{\mathbf{R}^{-i}}}$.

***Step 4*** *The rule F is strategy-proof.*

Let $\mathbf{R} \in \mathcal{R}^N$, $i \in N$. First, we prove that

$$\forall k \in \mathbb{N},\ F^i(\mathbf{R})\ R^i\ f^i_k(\mathbf{R}). \tag{4}$$

Because for each $k \in \mathbb{N}$, $B^{f^i_{k\mathbf{R}^{-i}}} \subseteq B^{f^i_{k+1\mathbf{R}^{-i}}}$, $f^i_{k+1}(\mathbf{R})\ R^i\ f^i_k(\mathbf{R})$. Since $R^i$ is continuous and $F^i(\mathbf{R}) = \lim_{k\to+\infty} f^i_{\mathbf{R}(k+1)}(\mathbf{R})$, we obtain (4).

Next, we prove $F \in \Gamma_{\mathrm{SP}}$. To this end, suppose the contrary. That is

$$\exists x^i \in B^{F^i_{\mathbf{R}^{-i}}} \text{ s.t. } x^i\ P(R^i)\ F^i(\mathbf{R}).$$

Since $x^i \in B^{F^i_{\mathbf{R}^{-i}}}$, for some $\hat{R}^i \in \mathcal{R}$, $F^i(\hat{R}^i; \mathbf{R}^{-i}) = x^i$. Let $\hat{\mathbf{R}} = (\hat{R}^i; \mathbf{R}^{-i})$. By the definition of $F$,

$$x^i = \lim_{k\to+\infty} f^i_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}}). \tag{5}$$

Since $x^i \in \mathrm{SUC}(R^i, F^i(\mathbf{R}))$ and $R^i$ is continuous, $B(x^i, \varepsilon) \subseteq \mathrm{SUC}(R^i, F^i(\mathbf{R}))$ for some $\varepsilon > 0$.[32] By (5), for sufficiently large $L \in \mathbb{N}$, $f^i_{\hat{\mathbf{R}}(L)}(\hat{\mathbf{R}}) \in \mathrm{SUC}(R^i, F^i(\mathbf{R}))$. By (4), $\mathrm{SUC}(R^i, F^i(\mathbf{R})) \subseteq \mathrm{SUC}(R^i, f^i_{\hat{\mathbf{R}}(L)}(\mathbf{R}))$. This implies that $f^i_{\hat{\mathbf{R}}(L)}(\hat{\mathbf{R}}) \in \mathrm{SUC}(R^i, f^i_{\hat{\mathbf{R}}(L)}(\mathbf{R}))$. This means that $f^i_{\hat{\mathbf{R}}(L)}$ is manipulable by $i$ at $\mathbf{R} = (R^i; \mathbf{R}^{-i})$ via $\hat{R}^i$. This contradicts the *SP* of $f^i_{\hat{\mathbf{R}}(L)}$. Hence $F \in \Gamma_{\mathrm{SP}}$.

By Step 4, $[F] \in \Gamma_{\mathrm{SP}}/\sim$. By Step 3 and the definition of $C^{\mathbf{R}^{-i}}$,

$$\forall [f] \in \mathcal{S},\ [F] \text{ Dom } [f].$$

Hence $[F]$ is an upper bound of $\mathcal{S}$. □

*Proof of Theorem 1* By Lemma 5 and Zorn's lemma, for each $f \in \Gamma_{\mathrm{SP}}$, there exists $f_0 \in \Gamma_{\mathrm{SP}}$ such that $f_0$ dom $f$ and $[f_0]$ is a maximal element of $\left(\Gamma_{\mathrm{SP}}/\sim, \mathrm{Dom}\right)$. □

[32] Let $\mathrm{UC}(R, x) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid y\ R\ x\}$, $\mathrm{SUC}(R, x) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid y\ P(R)\ x\}$ and $\mathrm{LC}(R, x) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid x\ R\ y\}$ for each $x \in \prod_{j=1}^m [0, \Omega_j]$, and each $R \in \mathcal{R}$.

*Proof of Theorem 2* Let $E$ be the rule defined by

$$\forall \mathbf{R} \in \mathcal{R}^N, \forall i \in N, E^i(\mathbf{R}) = \frac{\Omega}{n}.$$

Obviously $E$ satisfies *SP* and *ER*. By Theorem 1, there exists a rule $E_0$ which satisfies *WSESP* and $E_0$ dom $E$. Since dom is transitive, $E_0$ satisfies *ER*. □

### A.2 Proofs of Theorems 3 and 4

First, we introduce a useful lemma. A proof is found in Amorós (2002).

**Lemma 6** *If* $x^*, x', x'' \in \prod_{j=1}^m [0, \Omega_j]$ *are such that*

$$\left[\exists j \in M \text{ s.t. } |x_j^* - x_j''| < |x_j^* - x_j'|\right] \text{ or } \left[\exists j \in M \text{ s.t. } (x_j^* - x_j'')(x_j^* - x_j') < 0\right],$$

*then there exists* $R \in \mathcal{R}$ *such that* $p(R) = x^*$ *and* $x'' P(R) x'$.

We will find the following equivalent statement of Lemma 6 is convenient.

**Lemma 7** *If* $x^*, x', x'' \in \prod_{j=1}^m [0, \Omega_j] (x' \neq x'')$ *are such that*

$$\neg[\forall j \in M, x_j^* \leq x_j' \leq x_j'' \text{ or } x_j'' \leq x_j' \leq x_j^*],$$

*then there exists* $R \in \mathcal{R}$ *such that* $p(R) = x^*$ *and* $x'' P(R) x'$.

Before we prove Theorem 3, we establish the following lemma about the shape of option sets. Note that Lemma 8 holds for any number of agents.

**Lemma 8** *Suppose that* $f$ *satisfies SP and WP. Then,*

$$\forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, \forall j \in M, \exists a_j, b_j \in [0, \Omega_j] \text{ s.t. } B^{f^i_{\mathbf{R}^{-i}}} = \prod_{j=1}^m [a_j, b_j].$$

*Proof* Let $i \in N$ and $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$. The proof is in two steps.

***Step 1*** The option set $B^{f^i_{\mathbf{R}^{-i}}}$ is convex.

Suppose by contradiction, that

$$\exists \hat{v}, \hat{w} \in B^{f^i_{\mathbf{R}^{-i}}}, \exists \lambda \in (0, 1) \text{s.t.} \lambda \hat{v} + (1-\lambda)\hat{w} \notin B^{f^i_{\mathbf{R}^{-i}}}.$$

Obviously,

$$\exists v, w \in B^{f^i_{\mathbf{R}^{-i}}}, \forall \lambda \in (0, 1) \text{s.t.} \lambda v + (1-\lambda) w \notin B^{f^i_{\mathbf{R}^{-i}}}.$$

Let $\tilde{x} = \frac{1}{2}v + \frac{1}{2}w$ and let $\tilde{R} \in \mathcal{R}$ be a preference that satisfies $p(\tilde{R}) = \tilde{x}$.
*Case 1.* $f^i(\tilde{R}; \mathbf{R}^{-i}) = v$ or $f^i(\tilde{R}; \mathbf{R}^{-i}) = w$

For each $\mathbf{d} = (d_1, \ldots, d_m) \in \prod_{j=1}^m \{e_j, -e_j\}$ and each $y \in \prod_{j=1}^m [0, \Omega_j]$, $E(y, \mathbf{d}) := \{z \in \prod_{j=1}^m [0, \Omega_j] \mid \exists \gamma_1, \ldots, \gamma_m \in \mathbb{R}_+ \text{ s.t. } z = y + \sum_{j=1}^m \gamma_j d_j\}$, where $e_j$ denotes the $m$-dimensional vector whose $j$th coordinate is 1 and the other coordinates are 0. Without loss of generality, we may assume $f^i(\tilde{R}; \mathbf{R}^{-i}) = v$. Let $\mathbf{d} = (d_1, \ldots, d_m)$ and $\mathbf{d}' = (d'_1, \ldots, d'_m) \in \prod_{j=1}^m \{e_j, -e_j\}$ be such that $v \in E(p(\tilde{R}), \mathbf{d})$ and $w \in E(p(\tilde{R}), \mathbf{d}')$. Suppose also that $\mathbf{d}$ and $\mathbf{d}'$ are such that

$$\forall j \in M, \left[v \in E(p(\tilde{R}), (-d_j, \mathbf{d}_{-j})) \text{ or } w \in E(p(\tilde{R}), (-d'_j, \mathbf{d}'_{-j})) \Rightarrow d_j = d'_j\right], \tag{6}$$

where $(-d_j, \mathbf{d}_{-j}) = (d_1, \ldots, d_{j-1}, -d_j, d_{j+1}, \ldots, d_m)$ and $(-d'_j, \mathbf{d}'_{-j})$ is defined in the same manner. Obviously, because $p(\tilde{R})$ is the midpoint of $v$ and $w$, there exists $j' \in M$ such that $d_{j'} = -d'_{j'}$. Since $v_{j'} < \tilde{x}_{j'} < w_{j'}$ or $w_{j'} < \tilde{x}_{j'} < v_{j'}$, by Lemma 6, there exists $\hat{R}^i \in \mathcal{R}$ such that $p(\hat{R}^i) = \tilde{x}$ and $w \ P(\hat{R}^i) \ v$. Hence, by Lemma 3, $f^i(\hat{R}^i; \mathbf{R}^{-i}) \neq v$. Since $f$ satisfies *WP*, this is a contradiction.
*Case 2.* $f^i(\tilde{R}; \mathbf{R}^{-i}) \neq v$ and $f^i(\tilde{R}; \mathbf{R}^{-i}) \neq w$ Let $c = f^i(\tilde{R}; \mathbf{R}^{-i})$. If

$$[\exists j \in M \text{ s.t. } |\tilde{x}_j - v_j| < |\tilde{x} - c_j| \text{ or } (\tilde{x} - v_j)(\tilde{x} - c_j) < 0]$$

or

$$[\exists j \in M \text{ s.t. } |\tilde{x}_j - w_j| < |\tilde{x} - c_j| \text{ or } (\tilde{x} - w_j)(\tilde{x} - c_j) < 0],$$

then by Lemma 6,

$$\left[\exists \tilde{R}_v \in \mathcal{R} \text{ s.t. } p(\tilde{R}_v) = \tilde{x} \text{ and } v \ P(\tilde{R}_v) \ c\right] \quad \text{or}$$
$$\left[\exists \tilde{R}_w \in \mathcal{R} \text{ s.t. } p(\tilde{R}_w) = \tilde{x} \text{ and } w \ P(\tilde{R}_b) \ c\right].$$

This contradicts the fact that $c = f^i(\tilde{R}; \mathbf{R}^{-i})$ and $f$ is *weakly peak only*. Hence,

$$[\forall j \in M, \ |\tilde{x}_j - v_j| \geq |\tilde{x} - c_j| \text{ and } (\tilde{x} - v_j)(\tilde{x} - c_j) \geq 0]$$

and

$$[\forall j \in M, \ |\tilde{x}_j - w_j| \geq |\tilde{x} - c_j| \text{ and } (\tilde{x} - w_j)(\tilde{x} - c_j) \geq 0].$$

Suppose that $\mathbf{d}, \mathbf{d}' \in \prod_{j=1}^m \{e_j, -e_j\}$ satisfy $v \in E(p(\tilde{R}), \mathbf{d})$, $w \in E(p(\tilde{R}), \mathbf{d}')$ and condition (6) in *Case 1*. Let $j \in M$. If $d_j = -d'_j$, because $(\tilde{x} - v_j)(\tilde{x} - c_j) \geq 0$ and $(\tilde{x} - w_j)(\tilde{x} - c_j) \geq 0$ then $c_j = \tilde{x}_j$. If $d_j = d'_j$, there extst $\lambda$ and $\lambda'$ such that $v_j = \tilde{x}_j + \lambda d_j$ and $w_j = \tilde{x}_j + \lambda' d_j$. Since $\tilde{x} = \frac{1}{2}v + \frac{1}{2}w, \lambda = \lambda'$. Hence,

$v_j = w_j = \tilde{x}_j$. Since $|\tilde{x}_j - v_j| \geq |\tilde{x} - c_j|$, $|\tilde{x} - c_j| = 0$. Hence, $c_j = \tilde{x}_j$. We have shown that $c = \tilde{x}$. Since $\tilde{x} \notin B^{f^i_{\mathbf{R}^{-i}}}$ and $c \in B^{f^i_{\mathbf{R}^{-i}}}$, a contradiction.

***Step 2*** For each $j \in M$, there exists $a_j$ and $b_j \in [0, \Omega_j]$ s.t. $B^{f^i_{\mathbf{R}^{-i}}} = \prod_{j=1}^m [a_j, b_j]$.

For each $j \in M$, let $\Pr_j$ denote the projection with respect to $j$th coordinate. Since $\Pr_j$ is continuous and $B^{f^1_{R^2}}$ is compact, $\Pr_j(B^{f^i_{\mathbf{R}^{-i}}}) \subseteq [0, \Omega_j]$ is compact. Let $a_j = \min \Pr_j(B^{f^i_{\mathbf{R}^{-i}}})$ and $b_j = \max \Pr_j(B^{f^i_{\mathbf{R}^{-i}}})$. We show that $\prod_{j=1}^m \{a_j, b_j\} \subseteq B^{f^i_{\mathbf{R}^{-i}}}$. However, we only show that $(b_1, \ldots, b_m) \in B^{f^i_{\mathbf{R}^{-i}}}$. We can handle the other cases in the same manner. Suppose to the contrary, that $(b_1, \ldots, b_m) \notin B^{f^i_{\mathbf{R}^{-i}}}$. Let $\tilde{\tilde{R}} \in \mathcal{R}$ be a preference that satisfies $p(\tilde{\tilde{R}}) = (b_1, \ldots, b_m)$. Let $h \in B^{f^i_{\mathbf{R}^{-i}}}$ be such that $f^i(\tilde{\tilde{R}}; \mathbf{R}^{-i}) = h$. Since $h \neq (b_1, \ldots, b_m)$,

$$\exists j' \in M \text{ s.t. } h_{j'} < b_{j'}.$$

Since $b_{j'} = \max \Pr_j(B^{f^i_{\mathbf{R}^{-i}}})$, there exists $h' \in B^{f^i_{\mathbf{R}^{-i}}}$ such that $h'_{j'} = b_{j'}$. Hence, by Lemma 6, and since $|b_{j'} - h'_{j'}| < |b_{j'} - h_{j'}|$,

$$\exists R^i \in \mathcal{R} \text{ s.t. } p(R^i) = (b_1, \ldots, b_m) \text{ and } h' \; P(R^i) \; h.$$

However, this implies

$$f^i(R^i; \mathbf{R}^{-i}) \neq h,$$

a contradiction to *WP*. □

*Proof of Theorem 3* Let $f$ be a rule satisfying *SP* and *same-sidedness*. By Lemma 3 in Amorós (2002), $f$ satisfies *WP*. We concludes by contradiction. Suppose that there exists $g \in \Gamma_{\mathrm{SP}}$ such that $g$ dom $f$ but $f \neq g$. By Lemma 4, $B^{f^1_{R^2}} \subseteq B^{g^1_{R^2}}$ and $B^{f^2_{R^1}} \subseteq B^{g^2_{R^1}}$ for each $(R^1, R^2) \in \mathcal{R}^N$. We claim that

$$\exists (R^1, R^2) \in \mathcal{R}^N \text{ s.t. } B^{f^1_{R^2}} \subsetneq B^{g^1_{R^2}} \text{ or } B^{f^2_{R^1}} \subsetneq B^{g^2_{R^1}}. \tag{7}$$

If not, then $B^{f^1_{R^2}} = B^{g^1_{R^2}}$ and $B^{f^2_{R^1}} = B^{g^2_{R^1}}$ for each $(R^1, R^2) \in \mathcal{R}^N$. Note that by Lemma 8, $B^{f^1_{R^2}}$ is a direct product of closed intervals. By single-peakedness, $\#\tau(R^1, B^{f^1_{R^2}}) = 1$ for each $(R^1, R^2) \in \mathcal{R}^N$. Then, by Lemma 3, for each $(R^1, R^2) \in \mathcal{R}^N$, $f^1(R^1, R^2) = g^1(R^1, R^2)$. By feasibility, for each $(R^1, R^2) \in \mathcal{R}^N$, $f^2(R^1, R^2) = g^2(R^1, R^2)$. Hence, $U = g$, a contradiction. Hence (7) holds. Without loss of generality, suppose that there exists $R^2 \in \mathcal{R}$ such that $B^{f^1_{R^2}} \subsetneq B^{g^1_{R^2}}$.

We have $\tilde{x} \in \prod_{j=1}^m [0, \Omega_j]$ such that $\tilde{x} \in B^{g^1_{R^2}}$ and $\tilde{x} \notin B^{f^1_{R^2}}$. Let $\tilde{R} \in \mathcal{R}$ be such that $p(\tilde{R}) = \tilde{x}$. By Lemma 8, for each $j \in M$, there exist $a_j, b_j \in [0, \Omega_j]$ such that

$B^{f_R^i} = \prod_{j=1}^m [a_j, b_j]$. Then, for each $j \in M$, one of the following three holds;

$$\begin{aligned}
&\text{(i)} \quad \tilde{x}_j < a_j \quad (\Leftrightarrow \Omega_j - a_j < \Omega_j - \tilde{x}_j),\\
&\text{(ii)} \quad \tilde{x}_j \in [a_j, b_j] \quad (\Leftrightarrow \Omega_j - \tilde{x}_j \in [\Omega_j - b_j, \Omega_j - a_j]),\\
&\text{(iii)} \quad b_j < \tilde{x}_j \quad (\Leftrightarrow \Omega_j - \tilde{x}_j < \Omega_j - b_j).
\end{aligned}$$

Let $y \in B^{f^1_{R^2}}$ be defined as follows; for each $j \in M$, if (i) holds, then $y_j = a_j$, if (ii) holds, then $y_j = \tilde{x}_j$, and if (iii) holds, then $y_j = b_j$. Obviously, $\tau(\tilde{R}, B^{f^1_{R^2}}) = \{y\}$. Hence,

$$f^1(\tilde{R}, R^2) = y \text{ and } g^1(\tilde{R}, R^2) = \tilde{x}.$$

Now let us consider the allotment for agent 2. For each $Y \subseteq \prod_{j=1}^m [0, \Omega_j]$, define $sym(Y) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid \exists x \in Y \text{ s.t. } y = \Omega - x\}$. Let $R^* \in \mathcal{R}$ be such that $p(R^*) = \Omega - p(R^2)$. Then, by *same-sidedness*, $\Omega - p(R^2) = f^1(R^*, R^2)$. Hence, $\Omega - p(R^2) \in B^{f^1_{R^2}}$. Hence, $p(R^2) \in sym(B^{f^1_{R^2}}) = \prod_{j=1}^m [\Omega_j - b_j, \Omega_j - a_j]$. By the definition of $y$ and $\tilde{x}$, for each $j \in M$,

$$\begin{aligned}
\text{(i)} \Rightarrow \quad & \Omega_j - y_j = \Omega_j - a_j,\\
\text{(ii)} \Rightarrow \quad & \Omega_j - y_j = \Omega_j - \tilde{x}_j,\\
\text{(iii)} \Rightarrow \quad & \Omega_j - y_j = \Omega_j - b_j.
\end{aligned}$$

Since for each $j \in M$, $p_j(R^2) \in [\Omega_j - b_j, \Omega_j - a_j]$, then

$$\forall j \in M, p_j(R^2) \leq \Omega_j - y_j \leq \Omega_j - \tilde{x}_j \text{ or } \Omega_j - \tilde{x}_j \leq \Omega_j - y_j \leq p_j(R^2).$$

Because $\tilde{x} \neq y$, $\Omega - \tilde{x} \neq \Omega - y$. By the single-peakedness of $R^2$, $(\Omega - y)\ P(R^2)\ (\Omega - \tilde{x})$. By the feasibility,

$$f^2(\tilde{R}, R^2) = \Omega - y \text{ and } g^2(\tilde{R}, R^2) = \Omega - \tilde{x}.$$

However, since $g$ dom $f$, this is a contradiction. □

**Lemma 9** *Suppose that* $n = 2$. *Suppose also that* $f$ *satisfies SP, ER and WP. Then,* $U$ *dom* $f$.[33]

*Proof* We show that for each $i \in N$ and each $R \in \mathcal{R}$, $B^{f_R^i} \subseteq B^{U_R^i}$. Then, we obtain the conclusion by Lemma 4. Without loss of generality, suppose that $i = 1$. Let $R^2 \in \mathcal{R}$. Then, *ER* and the feasibility condition imply that $B^{f^1_{R^2}} \subseteq sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))$.

[33] In the problem of public goods provision, Moulin (1994) and Olszewski (2004) show that the serial mechanism dominates any other mechanisms satisfying a certain set of axioms. Lemma 9 has the same spirit as their results.

***Step 1*** $\forall \tilde{R}^2 \in \mathcal{R}, \Big[ p(\tilde{R}^2) = p(R^2)$ and $\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}) \subseteq \mathrm{UC}(R^2, \frac{\Omega}{2}) \Rightarrow B^{f^1_{R^2}} \subseteq sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))\Big]$.

Suppose not. Then, there is $\tilde{R}^2 \in \mathcal{R}$ such that $p(\tilde{R}^2) = p(R^2), \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}) \subseteq \mathrm{UC}(R^2, \frac{\Omega}{2})$ and $B^{f^1_{R^2}} \not\subseteq sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))$. Then, there exists a consumption bundle $x$ such that $x \in B^{f^1_{R^2}}$ and $x \notin sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))$. Obviously we can take $R_x \in \mathcal{R}$ such that $p(R_x) = x$. By Lemma 3, $f(R_x, R^2) = (x, \Omega - x)$. Since $B^{f^1_{\tilde{R}^2}} \subseteq sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))$, then $x \notin B^{f^1_{\tilde{R}^2}}$. Hence, $f(R_x, \tilde{R}^2) \neq (x, \Omega - x)$. Then, $f$ satisfies *WP*, a contradiction.

***Step 2*** $\forall x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))\backslash B^{U^1_{R^2}}, \exists \tilde{R}^2 \in \mathcal{R}$ s.t. $p(\tilde{R}^2) = p(R^2)$ and $\Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$.

For each $j \in M$, define

$$a_j = \begin{cases} \Omega_j - p_j(R^2) & \text{if } \frac{\Omega_j}{2} \leq p_j(R^2), \\ \frac{\Omega_j}{2} & \text{otherwise}, \end{cases} \qquad b_j = \begin{cases} \frac{\Omega_j}{2} & \text{if } \frac{\Omega_j}{2} \leq p_j(R^2), \\ \Omega_j - p_j(R^2) & \text{otherwise}, \end{cases}$$

then $B^{U^1_{R^2}} = \prod_{j=1}^m [a_j, b_j]$. Hence, $sym(B^{U^1_{R^2}}) = \prod_{j=1}^m [\Omega_j - b_j, \Omega_j - a_j]$.

Let $x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))\backslash B^{U^1_{R^2}}$. Since $\frac{\Omega}{2} \in B^{U^1_{R^2}}$, $x \neq \frac{\Omega}{2}$. Hence, $\Omega - x \neq \frac{\Omega}{2}$. We show the following by contradiction.

$$\neg \left[ \forall j \in M, \frac{\Omega_j}{2} \leq \Omega_j - x_j \leq p_j(R^2) \text{ or } p_j(R^2) \leq \Omega_j - x_j \leq \frac{\Omega_j}{2} \right]. \tag{8}$$

Suppose not. Then, for each $j \in M$, if $\frac{\Omega_j}{2} \leq \Omega_j - x_j \leq p_j(R^2)$, then $\Omega_j - p_j(R^2) \leq x_j$ and $x_j \leq \frac{\Omega_j}{2}$. This is equivalent to $x_j \in [a_j, b_j]$. Similarly we can show that for each $j \in M$, if $p_j(R^2) \leq \Omega_j - x_j \leq \frac{\Omega_j}{2}$, then $x_j \in [a_j, b_j]$. Hence we have shown that $x \in B^{U^1_{R^2}}$, a contradiction. We have (8).

By Lemma 7, there exists $\tilde{R}^2 \in \mathcal{R}$ such that

$$p(\tilde{R}^2) = p(R^2) \text{ and } \frac{\Omega}{2} \; P(\tilde{R}^2) \; (\Omega - x).$$

Now we show $B^{f^1_{R^2}} \subseteq B^{U^1_{R^2}}$ by contradiction. Suppose that we have a consumption bundle $x$ in $B^{f^1_{R^2}}$ but not in $B^{U^1_{R^2}}$. Then, *ER* and feasibility imply that $x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))$. By Step 2, there is a preference $\tilde{R}^2 \in \mathcal{R}$ such that $p(\tilde{R}^2) = p(R^2)$ and $\Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$. Let $R_x$ be a preference whose peak is $x$. Then, by Lemma 3 and feasibility, $f(R_x, R^2) = (x, \Omega - x)$. By *WP*, $f(R_x, \tilde{R}^2) = (x, \Omega - x)$. $f^2(R_x, \tilde{R}^2) = \Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$ but $f$ satisfies *ER*, a contradiction. □

**Lemma 10** *Suppose that $n = 2$. Suppose also that $f$ satisfies SY and WP. Then,*

$$\forall \mathbf{R} \in \mathcal{R}^N, \forall i, j \in N, [R^i = R^j \Rightarrow f^i(\mathbf{R}) = f^j(\mathbf{R})].$$

*Proof* We prove the conclusion by a contradiction. Suppose that $\mathbf{R} = (R^1, R^2) \in \mathcal{R}^N$ satisfies $R^1 = R^2$ and $f(\mathbf{R}) \neq (\frac{\Omega}{2}, \frac{\Omega}{2})$. By *SY*, $f^1(\mathbf{R})\ I(R^1)\ f^2(\mathbf{R})$. Hence, by single-peakedness of $R^1$,

$$\neg[\forall j \in M, p_j(R^1) \leq f_j^1(\mathbf{R}) \leq f_j^2(\mathbf{R}) \text{ or } f_j^2(\mathbf{R}) \leq f_j^1(\mathbf{R}) \leq p_j(R^1)].$$

By Lemma 7, there exist $\hat{R} \in \mathcal{R}$ such that $p(\hat{R}) = p(R^1)$ and $f^2(\mathbf{R})\ P(\hat{R})\ f^1(\mathbf{R})$. Define $\hat{\mathbf{R}} = (\hat{R}, \hat{R})$. By *WP*, $f(\hat{\mathbf{R}}) = f(\mathbf{R})$. But this contradicts the *SY* of $f$. □

**Lemma 11** *Suppose that $n = 2$. Suppose also that $f$ satisfies SP, SY and WP. Then, $f$ satisfies ER.*

*Proof* We prove the conclusion by a contradiction. Without loss of generality, suppose that $\mathbf{R} = (R^1, R^2) \in \mathcal{R}^N$ satisfies $\frac{\Omega}{2}\ P(R^1)\ f^1(\mathbf{R})$. By Lemma 10, $f(R^2, R^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$. Hence, $f^1(R^2, R^2)\ P(R^1)\ f^1(\mathbf{R})$. This violates *SP*. □

*Proof of Theorem 4* Obviously $U$ satisfies *ER* and *WP*. By Theorem 3, $U$ satisfies *SSESP* which is a stronger requirement than *WSESP*. Next, we show the converse. Suppose that $f$ satisfies *WSESP, WP* and *SY*. Note that by Lemma 11, $f$ satisfies *ER*. By Lemma 9, $U$ dom $f$. Since $f$ satisfies *WSESP*, $f$ dom $U$. Since $U$ satisfies *SSESP*, $f = U$. □

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